On Calibration of Stochastic and Fractional Stochastic Volatility Models

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We consider the risk-neutral stock price model

\[ dS_t = rS_t \, dt + \sqrt{v_t} S_t \, d\tilde{W}_t^S, \]

\[ dv_t = \kappa (\theta - v_t) \, dt + \sigma \sqrt{v_t} \, d\tilde{W}_t^v, \]

\[ d\tilde{W}_t^S \, d\tilde{W}_t^v = \rho \, dt, \]

with initial conditions \( S_0 \geq 0 \) and \( v_0 \geq 0 \), where

- \( S_t \) is the price of the underlying asset at time \( t \),
- \( v_t \) is the instantaneous variance at time \( t \),
- \( r \) is the risk-free rate,
- \( \theta \) is the long run average price variance,
- \( \kappa \) is the rate at which \( v_t \) reverts to \( \theta \) and
- \( \sigma \) is the volatility of the volatility.

\((\tilde{W}_t^S, \tilde{W}_t^v)\) is a two-dimensional Wiener process under the risk-neutral measure \( \tilde{P} \) with instantaneous correl. \( \rho \).

European call option price $C(S, v, t)$ can be expressed as:

$$C(S, v, t) = S - Ke^{-r\tau} \frac{1}{\pi} \int_{0+i/2}^{\infty+i/2} e^{-ikX} \frac{\hat{H}(k, v, \tau)}{k^2 - ik} dk,$$

where

$$\hat{H}(k, v, \tau) = \exp \left( \frac{2\kappa\theta}{\sigma^2} \left[ tg - \ln \left( \frac{1 - he^{-\xi t}}{1 - h} \right) + vg \left( \frac{1 - e^{-\xi t}}{1 - he^{-\xi t}} \right) \right] \right),$$

$X = \ln(S/K) + r\tau$

g = \frac{b - \xi}{2}, \quad h = \frac{b - \xi}{b + \xi}, \quad t = \frac{\sigma^2\tau}{2},$

$$\xi = \sqrt{b^2 + \frac{4(k^2 - ik)}{\sigma^2}},$$

$$b = \frac{2}{\sigma^2} \left( ik\rho\sigma + \kappa \right).$$
Optimization problem, nonlinear least squares:

$$\inf_\Theta G(\Theta), \quad G(\Theta) = \sum_{i=1}^{N} w_i | C_{i\Theta}(t, S_t, T_i, K_i) - C_{i}^{*}(T_i, K_i)|^2,$$

where

- $N$ denotes the number of observed option prices,
- $w_i$ is a weight,
- $C_{i}^{*}(T_i, K_i)$ is the market price of the call option observed at time $t$,
- $C_{i\Theta}$ denotes the model price computed using vector of model parameters.

For Heston SV model we have $\Theta = (\kappa, \theta, \sigma, \nu_0, \rho)$. 
Considered algorithms and their implementations

We tested

- **global optimizers:**
  - in MATLAB’s Global Optimization Toolbox:
    - genetic algorithm (GA) - function `ga()`
    - simulated annealing (SA) - function `simulannealbnd()`
  - from inberg.com:
    - adaptive simulated annealing (ASA)

- **local search method (LSQ):**
  - in MATLAB’s Optimization Toolbox: function `lsqnonlin()`,
    - Gauss-Newton trust region,
    - Levenberg-Marquardt,
  - in Microsoft Excel’s solver
    - Generalized Reduced Gradient method,

- combination of both approaches, see later.
Measured errors, considered weights

Maximum absolute relative error

\[ \text{MARE}(\Theta) = \max_i \frac{|C_i^\Theta - C_i^*|}{C_i^*} \]

and average of the absolute relative error

\[ \text{AARE}(\Theta) = \frac{1}{N} \sum_{i=1}^{N} \frac{|C_i^\Theta - C_i^*|}{C_i^*} \]

for \( i = 1, \ldots, N \). Let \( \delta_i > 0 \) denote the bid ask spread. We consider the following weights

- weight A: \( w_i = \frac{1}{|\delta_i|} \),
- weight B: \( w_i = \frac{1}{\delta_i^2} \),
- weight C: \( w_i = \frac{1}{\sqrt{\delta_i}} \).
Empirical results for Heston model on real market data

DATA:
- Market prices obtained on March 19, 2013 from Bloomberg’s Option Monitor for ODAX call options.
- We used a set of 107 options for 6 maturities.
- Volatility smile and term structure for DAX call options (sourced from Bloomberg Finance L.P.):
Calibration results

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Weight</th>
<th>AARE</th>
<th>MARE</th>
<th>$\nu_0$</th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma$</th>
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<td>A</td>
<td>1.25%</td>
<td>12.46%</td>
<td>0.02897</td>
<td>0.68921</td>
<td>0.10313</td>
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<td>0.09508</td>
<td>1.44249</td>
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</tbody>
</table>

* initial guesses obtained by deterministic grid;
Results for pair GA and LSQ in terms of absolute relative errors:
Results for pair GA and LSQ in terms of absolute relative errors:
We consider the risk-neutral stock price model with approximative fractional stochastic volatility (FSV)

\[ dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^S + Y_t S_t dN_t, \]

\[ dv_t = -\kappa (v_t - \bar{v}) dt + \xi v_t dB_t^H, \]

where

- \( \kappa \) is a mean-reversion rate,
- \( \bar{v} \) stands for an average volatility level,
- \( \xi \) is so-called volatility of volatility,
- \( (N_t)_{t \geq 0} \) is a Poisson process,
- \( Y_t \) denotes an amplitude of a jump at \( t \),
- \( (W_t^S)_{t \geq 0} \) is a standard Wiener process,
- \( (B_t^H)_{t \geq 0} \) is an approximative fractional process.

Approximative fractional process

Let

\[ B_t^H = \int_0^t (t - s + \varepsilon)^H - 1/2 dW_s, \]

where

- \( H \) is a long-memory Hurst parameter in general \( H \in [0, 1] \),
- \( \varepsilon \) is a non-negative approximation factor,
- \((W_t)_{t \geq 0}\) represents a standard Wiener process.

Long-range dependence of volatility if \( H \in (0.5, 1] \).
If \( \varepsilon > 0 \) then \( B_t^H \) is a semi-martingale.
Semi-closed form solution of the FSV model

European call option price $V(\tau, K)$ can be expressed as:

$$V(\tau, K) = e^{x_t} P_1(x_t, \nu_t, \tau) - e^{-r\tau} K P_2(x_t, \nu_t, \tau),$$

where for $n = 1, 2$

$$P_n = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{i\phi \ln(K)} f_n}{i\phi} \right] d\phi,$$

$$f_n = \exp \left\{ C_n(\tau, \phi) + D_n(\tau, \phi) \nu_0 + i\phi \ln(S_t) + \psi(\phi) \tau \right\},$$

$$C_n(\tau, \phi) = r\phi i\tau + \theta Y_n \tau - \frac{2\theta}{\beta^2} \ln \left( \frac{1 - gn e^{d_n \tau}}{1 - gn} \right),$$

$$D_n(\tau, \phi) = Y_n \left( \frac{1 - e^{d_n \tau}}{1 - gn e^{d_n \tau}} \right),$$

where all the unexplained terms follow...
For \( n = 1, 2 \)

\[
\psi = -\lambda i \phi \left( e^{\alpha J + \gamma_j^2/2} - 1 \right) + \lambda \left( e^{i \phi \alpha J - \phi^2 \gamma_j^2/2} - 1 \right)
\]

\[
Y_n = \frac{b_n - \rho \beta \phi i + d_n}{\beta^2}
\]

\[
g_n = \frac{b_n - \rho \beta \phi i + d_n}{b_n - \rho \beta \phi i - d_n},
\]

\[
d_n = \sqrt{(\rho \beta \phi i - b_n)^2 - \beta^2 (2u_n \phi i - \phi^2)},
\]

\[
\beta = \xi \varepsilon^{H-1/2} \sqrt{v_t}, \quad u_1 = 1/2, \quad u_2 = -1/2, \quad \theta = \kappa \bar{v},
\]

\[
b_1 = \kappa - (H - 1/2) \xi \varphi_t - \rho \beta,
\]

\[
b_2 = \kappa - (H - 1/2) \xi \varphi_t.
\]

Rather complicated formula, but still 'Heston-like'.
The vector of parameters to be optimized will be
\[ \Theta = (v_0, \kappa, \bar{v}, \xi, \rho, \lambda, \alpha_J, \gamma_J, H), \]
where

<table>
<thead>
<tr>
<th>( v_0 )</th>
<th>( \kappa )</th>
<th>( \bar{v} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>initial volatility</td>
<td>mean reversion rate</td>
<td>average volatility</td>
</tr>
<tr>
<td>( \xi )</td>
<td>( \rho )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>volatility of volatility</td>
<td>correlation coef.</td>
<td>Poisson hazard rate</td>
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<tr>
<td>( \alpha_J )</td>
<td>( \gamma_J )</td>
<td>( H )</td>
</tr>
<tr>
<td>expected jump size</td>
<td>variance of jump sizes</td>
<td>Hurst parameter</td>
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Empirical results for the FSV model on real market data

**DATA:**
- Market prices obtained on January 8, 2014 from Bloomberg’s Option Monitor for British FTSE 100 stock index call options.
- We used a set of 82 options for 6 maturities.
## Calibration results

<table>
<thead>
<tr>
<th>Model</th>
<th>Weights</th>
<th>Algorithm</th>
<th>AARE [%]</th>
<th>MARE [%]</th>
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<td><strong>FSV model</strong></td>
<td>A</td>
<td>GA+LSQ</td>
<td>2.34</td>
<td>20.53</td>
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<tr>
<td></td>
<td></td>
<td>SA+LSQ</td>
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<td>20.53</td>
</tr>
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<td>19.93</td>
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The best calibration result in terms of AARE.
Results for pair GA and LSQ in terms of absolute relative errors for weights $B$:

FSV model

Heston model
Conclusion

Heston model:

- optimization problem is non-convex and may contain many local minima,
- local search method without a good initial guess may fail to achieve satisfactory results,
- we set a fine deterministic grid for initial starting points,
- best result of a trust region minimizer for these points (AARE=0.58%, MARE=3.10%) is taken as a reference point for comparison of less heuristic and more efficient approaches,
- with GA+LSQ we were able to get close (AARE=0.65%, MARE=2.22%).

Conclusion continued

FSV model:
- a new 'Heston-like' semi-closed formula,
- first empirical calibration results,
- in some aspects better results than with Heston model.


Further issues:
- optimization techniques:
  - performance and accuracy improvements of Gauss-Newton trust-region methods,
  - variable metric methods for nonlinear least squares,
  - fine tuning the global optimizers.
- presented approaches:
  - calibration results with respect to exotic derivatives,
  - hedging under the FSV model,
  - large-scale parallel calibration of the models.