A Macro-Finance Model of the Term Structure: the Case for a Quadratic Yield Model

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Structure of the presentation

Structure of the presentation:

- Motivation
- Model Formulation
- Filtering & Estimation
- Conclusions & Results

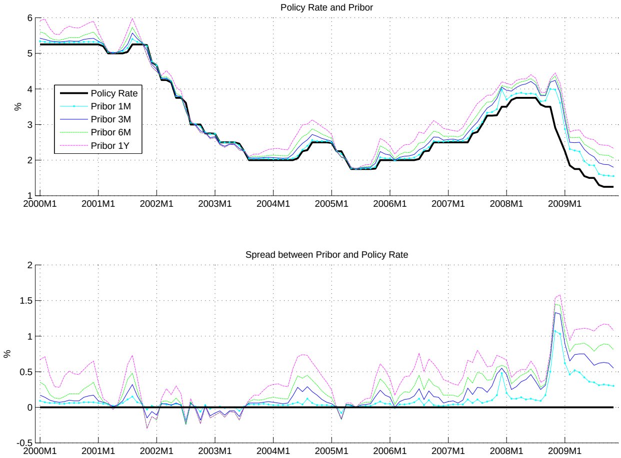
Macroeconomic models of the yield curve

- Add macro to the finance arbitrage-free models
  - affine models, (Ang-Piazzesi, 2003, 2006; Rudebusch-Wu, 2008);
  - Arbitrage-free Nelson-Siegel model (Christensen-Diebold-Rudebusch, 2008);
- Bond pricing in DSGE models (Rudebusch-Swanson-Wu, 2006);

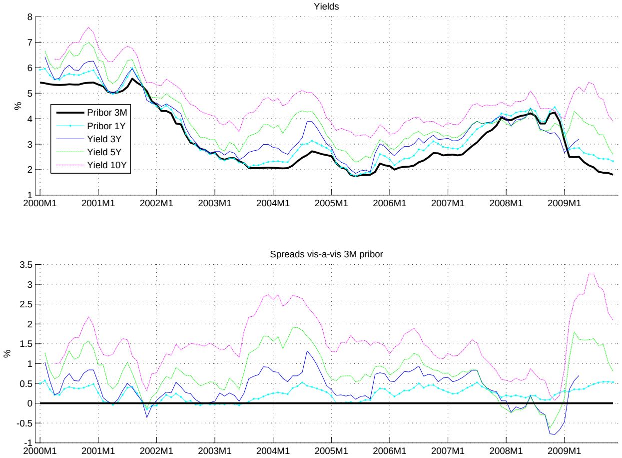
Applications:

- forecasting of inflation and real activity;
- central bank liquidity facilities;
- great moderation and great conundrum.

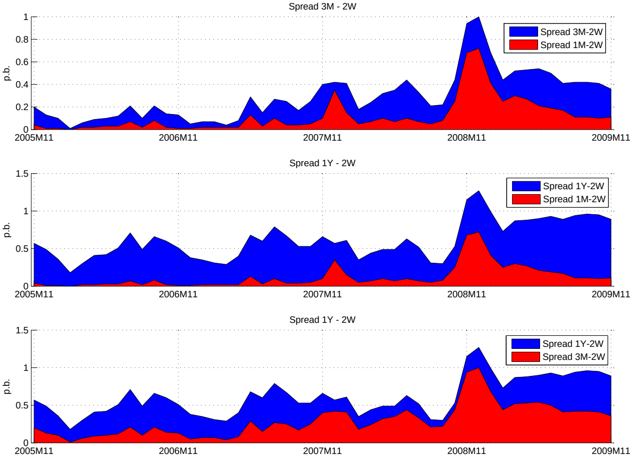
The spread on the money market



The spread between government bonds and the 3M Pribor



The decomposition of the spread



The Fama-Bliss regression

The FAMA-BLISS REGRESSION (AER, 1987):

$$\text{Excess Return}_{t+1} = \alpha + \beta \text{Spread}_t + \varepsilon_t.$$

- $\beta = 0$ is implied by the expectation hypothesis of the term structure.
- The recently introduced macro-finance term structure models (affine latent models) have problems if $\beta \neq 0$.

Econometric studies usually find that on the low end of the yield curve $\beta \cong 0$, i.e., the expectation hypothesis approximatively holds.

Results of the Fama-Bliss regression for the Czech money market

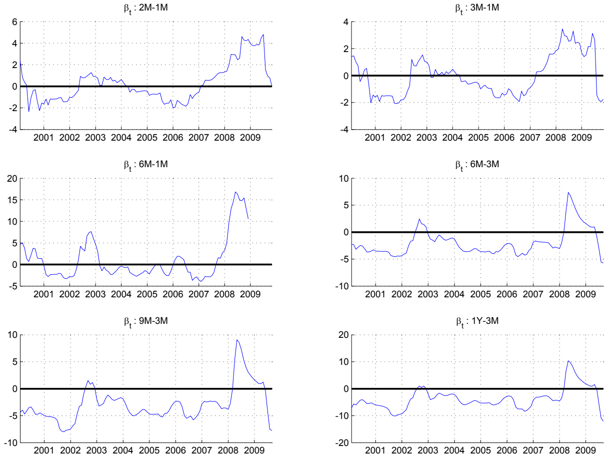
Rolling regression (18 months window) of the Fama-Bliss regression on the Czech data suggest that:

- up to the second half of 2008, $\beta \cong 0$
- $\beta \gg 0$ since then.

See next two figures:

1. Recursive estimation of the coefficient β for various maturities,
2. The dynamic prediction of the arbitrage-free version of the dynamic Nelson - Siegel model

Recursive estimation of Fama-Bliss βs



A Quadratic Term Structure Model

State equation

As in affine models, the underlying set of macroeconomic factors follows a VAR process:

$$X_{t+1} = \Phi X_t + \kappa + \Psi \varepsilon_{t+1}.$$

The log of the pricing kernel is defined as follows:

$$-m_{t+1} = \delta + \Gamma X_t + X_t^T \Delta X_t + \varsigma^t \varepsilon_{t+1} + \kappa \eta_{t+1},$$

hence the short-term interest rate i_t is given by:

$$i_t = -\mathbb{E}_t m_{t+1} - \frac{1}{2}\mathbb{V}_t m_{t+1} = \delta + \Gamma X_t + X_t^T \Delta X_t - \frac{1}{2}(\varsigma^T \varsigma + \kappa^2).$$

A Quadratic Term Structure Model – Recursion

The price of a k-period bond p_t^k is guessed in the form of:

$$-p_t^k = A_k + B_k^T + X_t^T C_k X_t,$$

with $A_1 = \delta - \frac{1}{2}(\varsigma^T \varsigma + \kappa^2)$, $B_1 = \Gamma$, $C_1 = \Delta$.

Under log-normality:

$$p_t^k = \mathbb{E}_t \left(m_{t+1} + p_{t+1}^{k+1} \right) + \frac{1}{2} \mathbb{V}_t \left(m_{t+1} + p_{t+1}^{k+1} \right),$$

$$i_t^k = -k^{-1} \log p_t^k.$$

A Quadratic Term Structure Model – Recursion (cont.)

The undetermined coefficient technique yields the following recursion:

$$-A_k = -\left(\delta + A_{k-1} + 2tr\left[C_{k-1}\Psi\Psi^T\right]\right) + \dots$$

$$\dots + \frac{1}{2}\left[\varkappa^2 + \varsigma^T\varsigma + B_{k-1}^T\Psi\Psi^T B_{k-1} + 2tr\left[\Psi^T C_{k-1}\Psi\Psi^T C_{k-1}\Psi\right]\right],$$

$$-B_k^T = -\left(\Gamma + B_{k-1}^T\Phi\right) + 2\left(\varsigma + B_{k-1}^T\Psi\right)\Psi^T C_{k-1}^T\Phi,$$

$$-C_k = -\left(\Delta + \Phi^T C_{k-1}\Phi\right) + 2\Phi^T C_{k-1}\Psi\Psi C_{k-1}^T\Phi.$$

Note that if $C_1 = \Delta$ is symmetric, so are C_k.

A Formulation of the Empirical Model

$$
\begin{aligned}
X_t &= \Phi X_{t-1} + \Psi \varepsilon_t, \\
i_t^k &= -\frac{A_k}{k} - \frac{B_k^T}{k} X_t + X_t^T \frac{-C_k}{k} X_t + \nu_{it}.
\end{aligned}
$$

We observe i_t^k and perhaps some elements of X_t.

A non-linear state space system has been obtained, we need to evaluate the likelihood (estimation), to filter the state (forecasting), and so on ....

Non-linear state space models

There are various choices of filtering non-linear state space models:

Extended Kalman filter: based on local linearization of the state and observation equations;

Gaussian sum filter: global filter; the probability distributions being approximated by a convex combination of gaussian pdfs;

Partcile filter: global filter; the distribution of states approximated using MC techniques;

Unscented filter: local filter based on unscented transformation.

The first-order approximation

Consider $X \sim N(\mu, \sigma^2)$ random variable and $Y = X^2$.

The first-order approximation (EKF) of Y works as follows:

$$\widetilde{Y}^1 = \mu^2 + \frac{\partial Y}{\partial X}|_{X=\mu}(X - \mu) = \mu^2 + 2\mu(X - \mu),$$

hence

$$\mathbb{E}\widetilde{Y}^1 = \mu^2 < \mathbb{E}Y = \mu^2 + \sigma^2$$
$$\mathbb{V}\widetilde{Y}^1 = 4\mu^2 \neq \mathbb{V}Y = 2\sigma^4 + \mu^2\sigma^2.$$

The first order approximation is biased

and the EKF may yield a biased estimation of the states.

The unscented filter

The unscented filter tries to approximate the mean and the variance of the non-linear transform more precisely than the first -order approximation.

Three possible ways:

1. unscented transform – a kind of quadrature;
2. Monte Carlo integration;
3. sometimes – exact integration possible.

Otherwise, the standard Kalman filter formulae apply. $\Longrightarrow$ thus this is an approximate filter, but hopefully more precise than the EKF.

The unscented transformation

How does the unscented transformation work?

1. Define a set of points: $\{x^{(\pm i)} | x^{(\pm i)} = \bar{x} \pm \Delta x^i\}$;
2. Compute $y^{(\pm i)} = f(x^{(\pm i)})$;
3. Set $\bar{y} = \sum_i w_i y^{(\pm i)}$; $P^y = \sum_i w_i (y^{(\pm i)} - \bar{y})(y^{(\pm i)} - \bar{y})^T$.

The unscented transformation $\Delta x^i = \sqrt{\frac{n}{1-w_0}} (P^x)_i^{1/2}$ and $w_i = \frac{1-w_0}{2n}$.

For large-dimensional problems, x^i could be randomly drawn from $N(0, P^x)$, and $w_i = n^{-1}$.

Algorithms' competition

I did a Monte Carlo study, which suggests that:

- the unscented filter can compete with the particle filter in precision,
- but is faster;
- the EKF and the Gaussian sum filter are slow and inaccurate.

Conclusion

It is feasible to estimate and filter the quadratic yield model in a reasonable time using the unscented filtering.