Lessons Learned from Stochastic Volatility Models Calibration and Simulation

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Outline

Stochastic volatility jump diffusion models
  Approximative fractional SVJD model

Lesson 1: Efficiency of pricing formulas

Lesson 2: Pitfalls of numerical integration

Lesson 3: Calibration of SV models

Lesson 4: Robustness and sensitivity analysis

Lesson 5: Simulation of SV models

Conclusion and further issues
We consider a general SVJD model which covers several kinds of stochastic volatility processes and also different types of jumps

\[ dS_t = (r - \lambda \beta)S_t dt + \sqrt{v_t}S_t dW^S_t + S_t dQ_t, \]

\[ dv_t = p(v_t)dt + q(v_t)dW^v_t, \]

\[ dW^S_t dW^v_t = \rho \ dt, \]

where \( p, q \in C^\infty(0, \infty) \) are general coefficient functions, \( r \) is the interest rate, \( \rho \) is the correlation of Wiener processes \( W^S_t \) and \( W^v_t \), parameters \( \lambda \) and \( \beta \) correspond to a specific jump process \( Q_t \), see below.

General model
Possible models

\[ dS_t = (r - \lambda \beta) S_t dt + \sqrt{v_t} S_t dW_t^S + S_t - dQ_t, \]
\[ dv_t = p(v_t) dt + q(v_t) dW_t^v. \]

<table>
<thead>
<tr>
<th>model</th>
<th>( p(v) )</th>
<th>( q(v) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heston/Bates</td>
<td>( \kappa(\theta - v) )</td>
<td>( \sigma \sqrt{v} )</td>
</tr>
<tr>
<td>3/2 model*</td>
<td>( \omega v - \tilde{\theta} v^2 )</td>
<td>( \xi v^{3/2} )</td>
</tr>
<tr>
<td>Geometric BM</td>
<td>( \alpha v )</td>
<td>( \xi v )</td>
</tr>
<tr>
<td>Fractional SVJD**</td>
<td>( (H - 1/2) \psi_t \sigma \sqrt{v} + \kappa(\theta - v) )</td>
<td>( \epsilon^{H - 1/2} \sigma \sqrt{v} )</td>
</tr>
</tbody>
</table>

\[ *\tilde{\theta} = -\frac{1}{2} \xi^2 + (1 - \gamma) \rho \xi \sqrt{(\theta + \gamma \frac{1}{2} \xi^2)^2 - \gamma (1 - \gamma) \xi^2}, \]

\[ **\psi_t = \int_0^t (t - s + \epsilon)^{H - 3/2} dW_s^\psi. \]
The jump process $Q_t$ is a compound Poisson process $Q_t = \sum_{i=1}^{N_t} Y_i$.

$Y_1, Y_2, \ldots$ are pairwise independent random variables with identically distributed jump sizes $\beta = \mathbb{E}[Y_i]$ for all $i \in \mathbb{N}$.

$N_t$ is a standard Poisson process with intensity $\lambda$ independent of the $Y_i$.

Jumps examples:

- log-normal, $\ln(1 + Y_i) \sim \mathcal{N}(\mu_J, \sigma_J^2)$, $\beta = \exp\{\mu_J + \frac{1}{2} \sigma_J^2\} - 1$.


- log-uniform, $\ln(1 + Y_i) \sim \mathcal{U}(a, b)$, $\beta = \frac{e^b-e^a}{b-a} - 1$.

  ![G. Yan and F.B. Hanson, Option Pricing for a Stochastic-Volatility Jump-Diffusion Model with Log-Uniform Jump-Amplitude, Proceedings of American Control Conference, 2006.]
The problem of pricing an option in a model with jumps corresponds to a partial integro-differential equation (PIDE). After substituting $x = \ln S$ we get the PIDE for 

$$f(x, v, t) = V(e^x, v, t)$$

$$-f_t = -rf + (r - \lambda \beta - \frac{1}{2}v)f_x + \frac{1}{2}vf_{xx} + pf_v + \frac{1}{2}q^2 f_{vv} + \rho q \sqrt{v} f_{xv}$$

$$+ \lambda \int_{-\infty}^{\infty} [f(x + y, v, t) - f(x, v, t)] \varphi(y) \, dy.$$


We can either solve the PIDE numerically or for simple contracts analytically - we can apply the complex Fourier transform similarly to what Lewis did for models without jumps

We want to apply the complex Fourier transform (Lewis, 2000)

\[ \mathcal{F} [f] = \hat{f}(k, v, t) = \int_{-\infty}^{\infty} e^{ikx} f(x, v, t) dx \]

with the inverse transform

\[ f(x, v, t) = \frac{1}{2\pi} \int_{-\infty+ik_i}^{\infty+ik_i} e^{-ikx} \hat{f}(k, v, t) dk \]

where \( k_i \) is some real number such that the line \((-\infty + ik_i, \infty + ik_i)\) is in some strip of regularity.

\[- \hat{f}_t = [-r - ik(r - \lambda \beta)] \hat{f} - \frac{1}{2} v(k^2 - ik)\hat{f} + (p - ikpq\sqrt{v})\hat{f}_v + \frac{1}{2} q^2 \hat{f}_{vv} \]

\[ + \lambda \mathcal{F} \left[ \int_{-\infty}^{\infty} [f(x + y, v, t) - f(x, v, t)] \varphi(y) dy \right]. \]
We have to derive the Fourier transform of the integral term

\[
\mathcal{F} \left[ \int_{-\infty}^{\infty} \left[ f(x + y, v, t) - f(x, v, t) \right] \varphi(y) dy \right] = \hat{f}(k, v, t)(\hat{\phi}(-k) - 1).
\]

We substitute \( \tau = T - t \) and define \( \hat{h}(k, v, t) \) by

\[
\hat{h}(k, v, t) = \exp \left( - \left[ -r - ik(r - \lambda \beta) + \lambda(\hat{\phi}(-k) - 1) \right] \tau \right) \hat{f}(k, v, \tau)
\]

and obtain the following equation

\[
\hat{h}_\tau = \frac{1}{2} q^2(v) \hat{h}_{vv} + \left[ p(v) - ikp(v)q(v)\sqrt{v} \right] \hat{h}_v - \frac{k^2 - ik}{2} v \hat{h}.
\]

We denote with \( \hat{H} \) the solution of the equation with initial value \( \hat{H}(k, v, 0) = 1 \) which is regular as a function of \( k = k_r + ik_i \) within a strip \( k_1 < k_i < k_2 \).
Our unifying formula for the European call price $V$ has the form

$$V(S, v, \tau) = S - Ke^{-r\tau} \frac{1}{2\pi} \int_{-\infty+ik_i}^{\infty+ik_i} e^{-ik\tilde{X}} e^{\lambda(\hat{\phi}(-k)-1)\tau} \frac{\hat{H}(k, v, \tau)}{k^2 - ik} dk,$$

where $\tilde{X} = \ln(S/K) + (r - \lambda\beta)\tau$ and $\max(k_1, 0) < k_i < \min(1, k_2)$.

<table>
<thead>
<tr>
<th>Financial claim</th>
<th>Payoff transform $\hat{w}(k)$</th>
<th>$k$-plane restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call option</td>
<td>$-\frac{K^{ik+1}}{k^2 - ik}$</td>
<td>$\Im k &gt; 1$</td>
</tr>
<tr>
<td>Put option</td>
<td>$-\frac{K^{ik+1}}{k^2 - ik}$</td>
<td>$\Im k &lt; 0$</td>
</tr>
<tr>
<td>Bull Spread option</td>
<td>$\frac{K_2^{ik+1} - K_1^{ik+1}}{k^2 - ik}$</td>
<td>$\Im k &gt; 0$</td>
</tr>
<tr>
<td>Bear Spread option</td>
<td>$\frac{K_1^{ik+1} - K_2^{ik+1}}{k^2 - ik}$</td>
<td>$\Im k &lt; 0$</td>
</tr>
<tr>
<td>Butterfly Spread</td>
<td>$\frac{2K_2^{ik+1} - K_1^{ik+1} - K_3^{ik+1}}{k^2 - ik}$</td>
<td>none</td>
</tr>
</tbody>
</table>
Let $B_t^\varepsilon = \int_0^t (t - s + \varepsilon)^{H - \frac{1}{2}} dW_s$ be the approximative fractional Brownian motion, $\varepsilon > 0$, $H > 0.5$ (for $H = 0.5$ it is the standard Brownian motion).

Then the volatility process in the approximative fractional SVJD model

$$d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dB_t^\varepsilon,$$

can be rewritten as

$$d\nu_t = \left[ (H - \frac{1}{2})\psi_t\sigma\sqrt{\nu_t} + \kappa(\theta - \nu_t) \right] dt + \varepsilon^{H - \frac{1}{2}}\sigma\sqrt{\nu_t}dW_t^\nu,$$

where $\psi_t = \int_0^t (t - s + \varepsilon)^{H - \frac{3}{2}} dW_s^\psi$.

We get the solution with

\[ V(S, v, \tau) = S - Ke^{-r\tau} \frac{1}{2\pi} \int_{-\infty+i/2}^{\infty+i/2} e^{-ikX} \frac{\hat{H}_f(k, v, \tau)}{k^2 - ik} \phi(-k) dk, \]

with \( \hat{H}_f(k, v, \tau) = \exp(C_f(k, \tau) + D_f(k, \tau)v) \) and

\[ C_f(k, \tau) = \kappa \theta Y \tau - \frac{2\kappa \theta}{B^2} \ln \left( \frac{1 - ge^{d\tau}}{1 - g} \right), \]

\[ D_f(k, \tau) = Y \frac{1 - e^{d\tau}}{1 - ge^{d\tau}}, \]

\[ Y = -\frac{k^2 - ik}{b - d}, \quad g = \frac{b + d}{b - d}, \quad d = \sqrt{b^2 + B^2(k^2 - ik)}, \]

\[ b = \kappa + ik\rho B, \]

\[ B = \varepsilon^{H - \frac{1}{2}} \sigma. \]
Lesson 1: Efficiency of pricing formulas

Computational efficiency of our solution for studied models:

- we compare computational time with respect to the original and newly proposed formulas,

- three pricing tasks - 100 European call options with different times to maturity and strike prices:
  1. 100 parameter sets - *market calibration with good initial guess*,
  2. 1000 parameter sets - *average calibration using local search method*,
  3. 10000 parameter sets - *calibration with global optimization procedure*.

- parameter sets are randomly generated in given parameter bounds.

Computation were made on a reference PC (2x Intel Xeon E5-2630 CPU and 12 GB RAM).
### Lesson 1: Efficiency of pricing formulas

Example: Bates model

<table>
<thead>
<tr>
<th>Pricing approach</th>
<th>Task</th>
<th>Time [sec]</th>
<th>Speed-up factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>#1</td>
<td>38.01</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>#2</td>
<td>407.16</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>#3</td>
<td>3396.74</td>
<td>-</td>
</tr>
<tr>
<td>Newly proposed</td>
<td>#1</td>
<td>9.37</td>
<td>4.06×</td>
</tr>
<tr>
<td></td>
<td>#2</td>
<td>80.98</td>
<td>5.03×</td>
</tr>
<tr>
<td></td>
<td>#3</td>
<td>926.10</td>
<td>3.67×</td>
</tr>
</tbody>
</table>


Lesson 1: Efficiency of pricing formulas

Bates model

Call prices of the Bates SVJD model

Differences between pricing approaches

Figure: Parameters: $\nu_0 = 0.025$, $\kappa = 0.98$, $\theta = 0.07$, $\sigma = 0.54$, $\rho = -0.65$, $\lambda = 0.5$, $\mu_J = -0.05$, $\sigma_J = 0.1$ for $S_0 = 100$, $\tau = 0.5$, $r = 0.03$. 
Lesson 1: Efficiency of pricing formulas
Comparison of the selected SVJD models

Figure: Option price as a function of the strike price for a call option with maturity 0.5 years and $S_0 = 100$, $r = 0.03$, $H = 0.7$. 
Lesson 2: Pitfalls of numerical integration

For some model parameters, we can observe (especially for adaptive quadrature algorithms):

▶ an enormous increase in function evaluations,
▶ serious precision problems as well as
▶ a significant increase in computational time.

Problems are caused by inaccurately evaluated integrands:

▶ all models (including Heston) affected,
▶ especially sensitive to the value of volatility of volatility $\sigma$,
▶ standard double vs. variable precision arithmetic (vpa).


Lesson 2: Pitfalls of numerical integration

FSV Example: Global view to integrated function

Figure: Parameters: $v_0 = 0.1$, $\kappa = 2.1$, $\theta = 0.4$, $\sigma = 0.002$, $\rho = -0.3$, $\lambda = 25$, $\mu_J = -4$, $\sigma_J = 1.7$, $H = 0.8$, for $S_0 = 6721.8$, $K = 6250$, $\tau = 0.120548$, $r = 0.009$. 
Lesson 2: Pitfalls of numerical integration

FSV Example: Detailed zoom of integrated function

Figure: Same parameters as before. Inaccurately enumerated values in standard double precision (red) and in vpa (blue) evaluated with 40 significant digits.
Optimization problem, nonlinear least squares:

\[ \inf_{\Theta} G(\Theta), \quad G(\Theta) = \sum_{i=1}^{N} w_i |C_{i}^{\Theta}(t, S_t, T_i, K_i) - C_{i}^{\ast}(T_i, K_i)|^2, \]

where

- \( N \) denotes the number of observed option prices,
- \( w_i \) is a weight,
- \( C_{i}^{\ast}(T_i, K_i) \) is the market price of the call option observed at time \( t \),
- \( C^{\Theta} \) denotes the model price computed using vector of model parameters.

Heston model: \( \Theta = (v_0, \kappa, \theta, \sigma, \rho) \),

Bates model: \( \Theta = (v_0, \kappa, \theta, \sigma, \rho, \lambda, \mu_J, \sigma_J) \),

Yan-Hanson model: \( \Theta = (v_0, \kappa, \theta, \sigma, \rho, \lambda, a, b) \),

FSV model: \( \Theta = (v_0, \kappa, \theta, \sigma, \rho, \lambda, \mu_J, \sigma_J, H) \).
Lesson 3: Calibration of SV models
Considered algorithms and their implementations

- **Global optimizers:**
  - In MATLAB's Global Optimization Toolbox:
    - **Genetic algorithm (GA)** - function `ga()`
    - **Simulated annealing (SA)** - function `simulannealbnd()`
  
  From `inberg.com`:
  - **Adaptive simulated annealing (ASA)**

- **Local search method (LSQ):**
  - In MATLAB's Optimization Toolbox: function `lsqnonlin()`,
    - **Gauss-Newton trust region,**
    - **Levenberg-Marquardt,**
  
  In Microsoft Excel's solver
  - **Generalized Reduced Gradient method,**

- Combination of both approaches, see later.
Maximum and average of absolute relative error

\[
\text{MARE}(\Theta) = \max_{i=1,...,N} \frac{|C_i^\Theta - C_i^*|}{C_i^*}, \quad \text{AARE}(\Theta) = \frac{1}{N} \sum_{i=1}^{N} \frac{|C_i^\Theta - C_i^*|}{C_i^*}
\]

Let \( \delta_i > 0 \) denote the bid ask spread.

We consider the following weights

\[
w_i^A = \frac{\frac{1}{N} \sum_{j=1}^{N} |\delta_j|^{-1}}{N \sum_{j=1}^{N} |\delta_j|^{-1}}, \quad w_i^B = \frac{\delta_i^{-2}}{N \sum_{j=1}^{N} \delta_j^{-2}}, \quad w_i^C = \frac{\delta_i^{-1/2}}{N \sum_{j=1}^{N} \delta_j^{-1/2}}
\]

\[
w_i^D = \frac{\text{Vega}_i^2}{N \sum_{j=1}^{N} \text{Vega}_j^2}, \quad w_i^E = \frac{1}{N}.
\]
Real market data

- option data really difficult to get for academic purposes,
- paid services such as Bloomberg Professional or Thomson Reuters Eikon rather expensive,
- exotic options almost impossible to get even with Bloomberg - only as OTC (private) contracts,
- a new cooperation with someone who can provide data welcomed.

We present an example:

- 97 ODAX calls traded on 18/03/2013 ranging from 86.5% to 112.0% moneyness across 5 maturities from ca 13.5 weeks to 1.76 years;
- 107 ODAX calls traded on 19/03/2013 ranging from 88.5% to 112.2% moneyness across 6 maturities from ca 13.4 weeks to 1.75 years.
Lesson 3: Calibration of SV models
Data source: Bloomberg Finance L.P.

Figure: Option price structure in the strike/maturity plane for ODAX call 18/03/2013 on the left and for 19/03/2013 on the right resp. The center of each circle corresponds to the strike/maturity parameters of the traded contract, circle diameter is proportionate to the option premium.
Lesson 3: Calibration of SV models

Data and figure source: Bloomberg Finance L.P.

Figure: Volatility smile and term structure for ODAX calls 19/03/2013.
Lesson 3: Calibration of SV models

Calibration results - SA vs. SA+LSQ

Figure: Calibration results for the FSV model using SA (left figure) and SA combined with LSQ.
Lesson 3: Calibration of SV models

Calibration results - GA+LSQ

Figure: Results of calibration for pair GA and LSQ for weights C - Heston model on the left and FSV model on the right.
Lesson 3: Calibration of SV models

Empirical results

- Optimization problem is non-convex and may contain many local minima,
- local search method without a good initial guess may fail to achieve satisfactory results,
- we can set a fine deterministic grid for initial starting points (rather time consuming, even in parallel environment), or we can use several iterations of a global optimizer (e.g. sufficiently large population in GA),
- Vega weights are least suitable.


Lesson 4: Robustness and sensitivity analysis

- New approach to robustness and sensitivity analysis for SV models is introduced,
- bootstrapping and Monte Carlo filtering techniques are applied,
- we address the impact of jumps and long memory in practice.

We present an example:

- European call options on AAPL. Four data sets from slightly different time periods: 01/04/2015, 15/04/2015, 01/05/2015 and 15/05/2015.
- Behavioural set of parameters - for which AARE is in lower 3/8 quantile, non-behavioural set - AARE in upper 3/8 quantile,

Table: Importance of $\lambda$ for calibrations AAPL options on all four datasets - for all we were able to reject the null hypothesis (both sets from the same distribution) at significance level 5%.

<table>
<thead>
<tr>
<th>Data sets</th>
<th>01/04/2015</th>
<th>15/04/2015</th>
<th>01/05/2015</th>
<th>15/05/2015</th>
</tr>
</thead>
<tbody>
<tr>
<td>p-value</td>
<td>1.30%</td>
<td>0.43%</td>
<td>8.45e-12%</td>
<td>3.56%</td>
</tr>
</tbody>
</table>

Scatterplot Matrix of Calibration Parameters in Bates Model

Figure: 15/05/2015. Diagonal elements depict histograms of parameter values obtained by bootstrap calibrations (e.g. the first histogram corresponds to the values of $\nu_0$). Off-diagonal elements illustrate a dependence structure for each parameter pair. In those figures, a black cross represents the reference value of the specific parameter (calibration to all data) and by a red star we depict the bootstrap estimate of the value.
Scatterplot Matrix of Calibration Parameters in FSV Model

- $v_0$
- $\kappa$
- $\theta$
- $\sigma$
- $\rho$
- $\lambda$
- $\mu_J$
- $\sigma_J$
- $H$

Parameter Ranges:
- $v_0$: 0.02 to 0.04
- $\kappa$: 0 to 100
- $\theta$: 0.04 to 0.08
- $\sigma$: 0 to 4
- $\rho$: -0.6 to -0.2
- $\lambda$: 0 to 0.4
- $\mu_J$: -10 to 10
- $\sigma_J$: 0 to 4
- $H$: 0.5 to 0.7

Graph axes display a grid of scatter plots comparing pairs of parameters.
Simulation of the CIR volatility process

- Euler scheme, Milstein scheme, absorption or reflection technique for positivity,
- exact scheme by Broadie and Kaya (2006) - problems with huge values of modified Bessel functions,
- QE scheme by Andersen (2008) - samples from approximated non-central chi-square distribution, probably the most efficient method.

For models with jumps a simple modification of the QE scheme is available.

For variance reduction we use antithetic variates method.

Lesson 5: Simulation of SV models

Example: Heston model simulation

Figure: Mean of 100,000 paths. Parameters: $v_0 = 0.02497$, $\kappa = 1.22136$, $\theta = 0.06442$, $\sigma = 0.55993$, $\rho = -0.66255$, $S_0 = 7962.31$, $r = 0.00207$, $T = 1$. Time step $\Delta = 2^{-6}$.
Lesson 5: Simulation of SV models

Example: Heston model simulation

![Graph showing convergence of schemes]

Figure: Convergence of schemes. 100,000 simulated paths, 100 batches. Parameters:

\[ \nu_0 = 0.02497, \kappa = 1.22136, \theta = 0.06442, \sigma = 0.55993, \rho = -0.66255, S_0 = 7962.31, \]

\[ r = 0.00207, T = 1. \] Time step \( \Delta = 2^{-4}, \ldots, 2^{-11} \).
Conclusion and further issues

FSV model:

▶ a new semi-closed formula,
▶ first empirical calibration results,
▶ in some aspects better results than with Heston model.

Further issues:

▶ performance and accuracy improvements of Gauss-Newton trust-region methods,
▶ variable metric methods for nonlinear least squares,
▶ fine tuning the global optimizers.
▶ efficient pricing of exotic derivatives,
▶ hedging under the FSV model,
▶ large-scale parallel calibration of the models.
Conclusion and further issues

- We are analyzing the PIDE that determines the Fundamental solution. What assumptions do we have to make on $p$ and $q$ to get:
  - existence and uniqueness of the solution or even an explicit solution,
  - certain degree of *regularity* for the solution, especially with respect to $k$ on the line $k = k_r + ik_i$ with $k_r \in \mathbb{R}$ and fixed $k_i \in \mathbb{R}$.

- We are studying the numerical integration techniques of the pricing formulas. There are many nontrivial open issues involving both the accuracy and the speed of calculation, some of them can be solved using the `vpa` only.

- For non-European type of contracts we are studying a numerical solution of the corresponding PIDE using finite difference methods and *finite element methods*.

- We are studying models with rough volatility, i.e. with driving process being not only approximative fractional, but for example standard *fractional Brownian motion*. 
Thank you for your attention!