SOLVING DYNAMIC ECONOMIC MODELS WITH RATIONAL EXPECTATIONS: A BRIEF EXPOSITION FOR NON-ECONOMISTS

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Abstract

Conditional expectations of future dated variables appear in economic models as a consequence of rational expectations (RE) hypothesis. Economic models, even linear, require then a special treatment with some new principles introduced in addition to the very classic methods of solving difference/differential models.

The paper outlines these principles using a simple example and briefly describes the Czech National Bank's toolbox for solving RE models.

Introduction

Expectations, i.e. subjective predictions of future dated variables, play a major role in economic (macroeconomic, economic policy, financial) modelling because economic agents' ideas about tomorrow affect to a substantial extent their today's behaviour. Not surprisingly, expectations formation has been at the heart of macroeconomic research for several decades. Nowadays the most accepted and broadly used concept of modelling expectations relies upon rationality hypothesis: economic agents (households, firms, governments, banks, investors, etc.) make use of all relevant information available (on relevant current and past variables as well as on the structure of economy) when doing their predictions. In other words, their subjective expectations (prediction of x_{t+k} made at t) are equal to (methematical) expectation conditional upon relevant information set Ω_t, i.e. $E(x_{t+k}|\Omega_t)$ which is throughout the paper abbreviated as $E_t\, x_{t+k}$.

In formal quantitative models, this type of expectations is generally treated as so-called *model consistent* whereby the expectation is set to the deterministic projection of the model. It means we let the information set contain only present and past values of variables included into the model and simplify the whole economy's structure to the model structure.

Solution to such dynamic models are distinguished from those to conventional models by the fact that they are not recursive in time: behaviour in the current period (current values of model variables) depends on the expected outcome for future periods, in addition

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to the usual dependency on past periods. This property brings an extra burden into the classic solution methods.

Successive section presents a very simple example to illustrate elementary rules for solving models with RE. Then a more general treatise is added for a certain (yet limited) class of models, and the paper is concluded with a brief description of the Czech National Bank's solution toolbox used for predictions and monetary policy analyses and simulations.

Readers should bear in mind that this paper is intended to be an intuitive illustration of solution technique ideas and its implementation in Matlab rather than a rigorous formal treatment.

Illustrative example

Let us start with a primitive model of financial market, more precisely, stock price model. We suppose two kinds of assets exist, interest-bearing bonds and stocks (shares in private companies). For the sake of simplicity the one-period rate of yield on bonds is assumed time-invariant and given, for instance, by the monetary policy, $i_t = i$.

Yield on stocks consists of two parts: change in price (spread between buying and selling price) and dividend paid to stockholders (i.e. shares in companies' profit). Then the one-period rate of yield on stocks is $(p_{t+1} + d_{t+1} - p_t)/p_t$, i.e. a stock is bought at time t at price p_t, in the next period the stockholder is paid dividend and sells the stock at p_{t+1}.

Finally, we assume *perfect substitutability* of bonds and stocks, and perfect flexibility of financial market. These imply that at each time t, the expected rate of yield on bonds and that on stocks must be equalized (so-called *no-arbitrage* condition): if this condition failed to hold, e.g. if expected rate of yield on stocks exceeded that on bonds, all investors would shift their demand to the more valuable kind of assets (stocks), which would in turn lead to a decline in its rate of yield (by virtue of elementary economic laws) and resume the equilibrium.

Given the interest rate i and provided investor's expectations are formed rationally (in terms of the previous section) we wish to determine development of the stock price based on the no-arbitrage condition

$$i = \frac{E_t\, p_{t+1} + E_t\, d_{t+1} - p_t}{p_t} \tag{1}$$

or

$$E_t\, p_{t+1} - j\, p_t + E_t\, d_{t+1} = 0, \qquad j = 1 + i > 1 \tag{2}$$

in other words to find a sequence $\{p_t\}$ of functions of variables in information set Ω_t which satisfies (2) for all possible realizations of these variables. Bear in mind that for our purposes we treat expected dividend stream as exogenously determined process so that

$$\Omega_t = \{p_t, p_{t-1}, \dots, j, d_t, d_{t-1}, \dots, E_t\, d_{t+1}, E_t\, d_{t+2}, \dots\} \tag{3}$$

With current period denoted $t = 0$, first step is to take expectations of (2) conditional upon Ω_0,

$$p^e_{t+1} - j\, p^e_t + d^e_{t+1} = 0, \qquad t = 0, 1, \dots \tag{4}$$

where $p^e_t = E_0\, p_t$ and $d^e_t = E_0\, d_t$.

The common solution to (4) is written as

$$p^e_t = j^t\, p^e_0 + \sum_{k=1}^{t} j^k\, d^e_k \tag{5}$$

with p^e_0 being the initial condition.

In our setup, however, we must realize that *no initial condition* p^e_0 is at hand because

- model (4) holds for $t = 0, 1, \dots$ only, not for $t < 0$; argumentation evolves from realizing that eq. (4) written at $t = -1$ gives

  $$E_0\, p_0 - j\, E_0\, p_{-1} + E_0\, d_0 = 0 \tag{6}$$

  or, equivalently

  $$p_0 - j\, p_{-1} + d_0 = 0 \tag{7}$$

  which is not what (2) actually says. Any past information is thus irrelevant for current and future stock price development.

- value of $p^e_0 = E_0\, p_0 = p_0$ or that of any later date is unavailable yet; search of their determination is what we are currently doing.

We arrived at the point where an infinite number of solutions exists. To discriminate among them and to choose one unique solution we recall that $j > 1$, thus all but one (provided process d^e_t is "sufficiently" stable, see Blanchard and Kahn (1980) or Klein (2000) for formal definitions) solutions are explosive in the sense p^e_t and consequently p_t blow up beyond all bounds as $t \to \infty$. Such an explosive behaviour called "bubble" or "sunspot" is observed in real economies (at financial markets expecially: demand for and thus the current price of stocks rises because it is expected to do so in the future bearing a higher yield) though it is rather an exception than a regularity. For this reason attention is paid to the only non-exploding solution in economics: it is often referred to as *fundamental* or *saddle-path* solution.

It is then easy to show what price $p^e_0 = p_0$ corresponds to this saddle-path condition. Iterating (4) forward we get

$$p^e_0 = \lim_{t \to \infty} \frac{1}{j^t}\, p^e_t + \sum_{k=1}^{\infty} \frac{1}{j^k}\, d^e_t \tag{8}$$

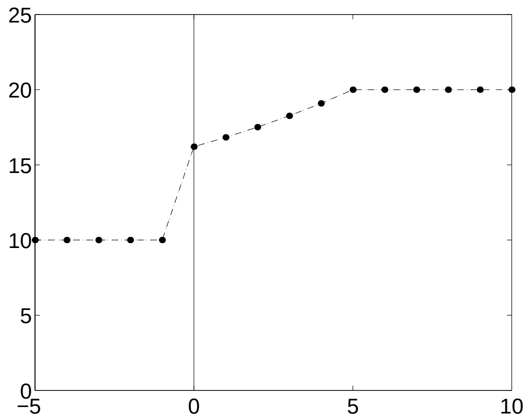

Figure 1: Stock price reaction to a new information

The limit is zero provided our saddle-path solution is "sufficiently" stable (see above references again) which in turn relies upon properties of the process d_t^e. Therefore under additional realistic assumptions we may write

$$p_0^e = p_0 = \sum_{k=1}^{\infty} \frac{1}{j^k} \, d_t^e \tag{9}$$

Current stock price equals to the discounted value (or "present value") of all expected future dividend paid on this stock.

It is worth noting that the necessary condition for existence of a unique stable (saddle-path) solution, namely $j > 1$ (or $|j| > 1$ in general), is met automatically in our setup: nominal interest rate i can never fall below zero and it is even highly unprobable for it to approach zero.

To conclude this section we illustrate an implication of rational expectations schema for model properties: any new information, even if it relates to some future event, is reflected immediately at the date of its release. Let us suppose following parametrization of (2): $j = 1.1$, $E_t \, d_{t+k} = 1, t < 0, \forall k$. Then, at $t = 0$ a new information occurs which leads to in increase in expected dividend paid on stocks at date $t = 5$ onward by 1 (e.g. a firm announces a long-run investment into its technology which is expected to improve firm's profit five years later), and this actually comes true). Formally,

$$d_t = \begin{cases} 1, & t < 5 \\ 2, & t \geq 5 \end{cases}$$

$$E_t \, d_{t+k} = 1 \quad \forall k, \quad t < 0$$

$$E_t \, d_{t+k} = \begin{cases} 1, & k < 5 \\ 2, & k \geq 5 \end{cases} \qquad t \geq 0$$

See Fig. 1 for reaction of stock price to the new information. At the date of release we observe a jump followed by a smooth development.

More general case

When dealing with more complex models where RE occur it is convenient to re-write them into a quasi-AR(1) representation

$$A\,E_t \begin{bmatrix} X_{t+1} \\ P_{t+1} \end{bmatrix} = B\,\begin{bmatrix} X_t \\ P_t \end{bmatrix} + C\,Z_t \tag{10}$$

where all endogenous variables are divided into subvectors X_t and P_t so that n-dimensional X_t contains all so-called *predetermined* or *backward-looking* variables satysfying

$$E_t\,X_{t+1} = X_{t+1} \tag{11}$$

(lags of endogenous variables or static transformations included in the model, typically), in contrast to *unpredetermined* or *forward-looking* variables comprised in m-dimensional vector P_t whose conditional expectations are involved in model equations. Z_t is an exogenously determined k-dimensional vector process or/and vector of random disturbances, A, B, C are known parameter matrices.

We sketch solution to (10) for the case A is regular. If it fails to be (which is much more frequent case in economic models) generalized Schur form and generalized eigenvalues concept must be employed as proposed by Klein (2000) instead of Jordan canonical form as below; though, the principles remain the same. We proceed analogously to (4): after premultiplying both sides of (10) by A^{-1} and taking expectations conditional upon information held at current date (say $t = 0$), Ω_0, we obtain

$$\begin{bmatrix} X^e_{t+1} \\ P^e_{t+1} \end{bmatrix} = \Gamma\,\begin{bmatrix} X^e_t \\ P^e_t \end{bmatrix} + \Phi\,Z^e_t, \quad t = 0, 1, \ldots \tag{12}$$

where $\Gamma = A^{-1}\,B$, $\Phi = A^{-1}\,C$, $X^e_t = E_0\,X_t$, $P^e_t = E_0\,P_t$, $Z^e_t = E_0\,Z_t$.

It is clear that only n initial conditions X^e_0 are now available (even if P_0 remains unpredetermined yet, $E_0\,X_0 = X_0$ is now condional upon Ω_{-1} solely). The model (10) has therefore a unique stable (saddle-path) solution if and only if matrix Γ has exactly m unstable eigenvalues (i.e. eigenvalues greater than one in absolute value) and exactly n stable ones (less or equal to one in absolute value). This is the celebrated Blanchard-Kahn (1980) saddle-path condition.

If there are more unstable eigenvalues than the number of unpredetermined variables is, then no stable solution exists. If there are less unstable eigenvalues than the number of unpredetermined variables is, then an infinite number of stable solutions exists.

Solution to (10) provided Blanchard-Kahn condition is satisfied is then found as follows:

1. Transform Γ into Jordan canonical form, $\Gamma = V^{-1}JV$, so that all n stable eigenvalues go first.

2. Decompose J, V, and $W \equiv V^{-1}$ into 4 blocks, the upper-left one being $(n \times n)$-dimensional, the lower-right one being $(m \times m)$-dimensional,

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \quad \text{etc.}$$

($J_{12} = J_{21} = 0$, obviously, and J_{11} is related to all stable eigenvalues whereas J_{22} to unstable ones). In adition, decompose

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$$

so that Φ_1 is an $(n \times k)$-dimensional block, Φ_2 is an $(m \times k)$-dimensional block.

3. Transform the original vector of endogenous variables into

$$\begin{bmatrix} S_t \\ U_t \end{bmatrix} = V \begin{bmatrix} X_t^e \\ P_t^e \end{bmatrix}$$

where S_t is an m-dimensional vector of transformed variables pertaining to stable eigenvalues, and P_t, n-dimensional, pertains to unstable ones.

4. Solve for the unstable part of the model, so that U_t is iterated forward and the unique non-explosive solution is chosen. This gives

$$U_0 = \sum_{k=0}^{\infty} J_{22}^{-k-1} \left(V_{21}\, \Phi_1 + V_{22}\, \Phi_2 \right) Z_k^e \tag{13}$$

5. Compute the initial condition S_0 recognizing

$$X_0 = B_{11}\, S_0 + B_{12}\, U_0 \tag{14}$$

where both X_0 and U_0 are already determined.

6. Transform S_0, U_0 back into X_0, P_0.

If Z_t consists only of random disturbances with known mean and of a constant term, the solution for X_t, P_t may be written in a vector ARMA representations with no expectational terms involved.

Matlab-based solution

Current version of Linda, Czech National Bank's own toolbox for solving, simulating and predicting linear dynamic models with rational expectations, is capable to deal with a large class of linear stochastic discrete-time models involving conditional expectations, including some non-trivial cases as

- models with past expectations of present values,

- models with integrated (unit-root) processes,
- models with cointegrated processes.

Processing of a model includes:

- Reading text-oriented, human-readable model description (definition of variables, parameters, list of model equations), its transformation into matrix form, detecting forward-looking and backward-looking variables, and checking the Blanchard-Kahn saddle-path condition.
- Solving the model, i.e. finding its ARMA representation with no expectational terms.
- Detecting integrated processes, detecting cointegrated processes and calculating cointegrating vectors.
- Simulating model response to a certain random disturbance hitting the system at present date or at a future date (so-called unexpected and expected shocks, in terminology of economic modelling).
- Predicting future actual development of the modelled system based on actual initial condition.
- Calculating autocovariance function of vector of endogenous variables (or their cointegrating combination, if any exist), based on the ARMA representation.
- Minimum variance control, i.e. optimization of some transformation of unconditional covariance matrix with respect to a given set of model parameters.

Interested readers will be given all m-files of current version of Linda (independent of any other Matlab toolboxes) at request by e-mail (see front page).

References

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