MODIFICATION OF ABEL TRANSFORMATION TO REFRACTIONAL METHODS

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Refractional methods enable visualization of inner conditions in transparent objects like gas flow or plasma discharge. Schlieren and shadow refractional methods depend on first or second derivatives of refractive index, respectively. As the derivatives of axially symmetric function violate its symmetry, the inverse Abel transformation cannot be applied directly to images of axially symmetric processes. We have adapted the Abel transformation to the refractional methods and on the basis of this formalism developed a code written in MATLAB, which for defined axially symmetric conditions reconstructs corresponding images.

1. Introduction

Interferometric and refractional methods are used to visualize inner conditions in transparent object, e.g. gas stream or low-temperature plasma. Both sorts of these methods are based on the dependence of the refractive index on an investigated quantity like electron number density, gas temperature, pressure or mass density [1,2]. In flowing gas the dependence of refractive index n on the gas mass density ρ is approximated by the Gladstone-Dale formula

$$\frac{n-1}{\rho} = \frac{n_0 - 1}{\rho_0} \tag{1}$$

where quantities denoted by zero subscript are related to a reference homogeneous state.

The interferometric methods [3,4] are recording the phase shift obtained by the ray passing through the medium. As the phase shift depends directly on the refractive index, this method is suitable for quantitative analysis, mainly for axially symmetric processes, in which case the inverse Abel transformation may be applied [7].

The refractive methods [1,2] are based on the angle deflection of the ray. As the effect of refraction depends on the first or even second derivatives of the refractive index with respect to spatial coordinates, the quantitative analysis is very complicated. Moreover, the derivatives disturb the axial symmetry and Abel transformation cannot be applied directly. This is the reason why the refractive methods are usually used just as a tool for understandable visualization of inner conditions, without quantitative processing.

In the diaphragm **schlieren method** the narrow light beam after its passage through the investigated medium is partially screened by the diaphragm. Negligible deflections above or below the diaphragm edge cause observable changes in the illumination of the screen. This variance enables to determine deflection angles of the ray. With *z*-axis oriented along the beam, the angular deflections in perpendicular *x*- and *y*- directions are

$$\alpha(x,y)=\frac{1}{n_0}\int_{z1}^{z2}\frac{\partial n(x,y,z)}{\partial x}\,\mathrm{d}z,\qquad \beta(x,y)=\frac{1}{n_0}\int_{z1}^{z2}\frac{\partial n(x,y,z)}{\partial y}\,\mathrm{d}z \tag{2}$$

respectively.

The **shadow** method is based on the phenomenon that the beam of the originally parallel rays becomes in an inhomogeneous medium convergent or divergent. The corresponding places on the projection screen become brighter or darker. Their intensity is approximately expressed as [1]

$$I(x,y)=\frac{I_0}{1+L\,J(x,y)} \tag{3}$$

where I_0 is the intensity for the homogeneous state, L is the distance of the screen from the place where the beam exits the medium, and

$$J(x,y)=\frac{1}{n_0}\int_{z1}^{z2}\left(\frac{\partial^2 n(x,y,z)}{\partial x^2}+\frac{\partial^2 n(x,y,z)}{\partial y^2}\right)\mathrm{d}z \tag{4}$$

In further considerations we replace refractive index by its relative change,

$$a=\frac{n-n_0}{n_0} \tag{5}$$

Hence,

$$\alpha(x,y)=\int_{z1}^{z2}\frac{\partial a}{\partial x}\mathrm{d}z,\qquad \beta(x,y)=\int_{z1}^{z2}\frac{\partial b}{\partial y}\mathrm{d}z \tag{6}$$

where we put with high accuracy $n_0\approx 1$.

2. Abel transformation and its reformulation

The Abel integral transformation and its inversion are most often met in connection with axially symmetric functions [4]. The ‘side-on’ function $g(x)$, e.g. light intensity or phase shift, is related to its radial source, e.g. refractive index, through the integral transformation and its inversion,

$$g(x)=2\int_{|x|}^{R} f(r)\,\frac{r\,\mathrm{d}r}{\sqrt{r^2-x^2}},\qquad f(r)=-\frac{1}{\pi}\int_{r}^{R}\frac{\mathrm{d}g(x)}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\sqrt{x^2-r^2}} \tag{7}$$

where R is the radius, outside which $f=g=0$. We will equate $R=\infty$.

To express the Abel transformation and its inverse counterpart in more compact form, we introduce new variables $\xi=x^2$, $\eta=r^2$ and define ‘Abel image’ $F(\xi)$ of the ‘Abel object’ $f(\eta)$ by

$$F(\xi) \equiv \int_{\xi}^{\infty} f(\eta)\sqrt{\eta - \xi}\,\mathrm{d}\eta \tag{8}$$

It is easy to verify following identities

$$\frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi} = -\frac{1}{2}\int_{\xi}^{\infty} \frac{f(\eta)}{\sqrt{\eta-\xi}}\mathrm{d}\eta = \int_{\xi}^{\infty} \frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\sqrt{\eta-\xi}\;\mathrm{d}\eta \tag{9}$$

The last equality, obtained by integration by parts, expresses that the derivative of an object is imaged into the derivative of its image.
Taken into account the last identities, we express the integral transformations (7) as

$$g(\xi) = -2\frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi}, \qquad f(\eta) = \frac{2}{\pi}\frac{\mathrm{d}^2 G(\eta)}{\mathrm{d}\eta^2} \tag{10}$$

where F and G denote images to f and g, respectively, as they are defined in (8).

3. Application of Abel transformation to refractive methods

We will consider axially symmetric processes in gas or plasma, with the symmetry axis oriented in y-direction. The refractive index n is then a function of the radial distance $r = \sqrt{x^2 + z^2}$ and the 'height' y.

3.1 Schlieren method

Assuming $a = a(r, y)$, the first formula in (6) may be rewritten as

$$\alpha(x, y) = 2x\int_{|x|}^{\infty} \frac{1}{r}\frac{\partial a}{\partial r}\frac{r\,\mathrm{d}r}{\sqrt{r^2 - x^2}} \tag{11}$$

Using coordinates ξ, η and denoting $\alpha(x, y)/x = g(\xi, y)$, we obtain

$$g(\xi, y) = 2\int_{\xi}^{\infty} \frac{\partial a/\partial\eta}{\sqrt{\eta-\xi}}\mathrm{d}\eta = -4\,\frac{\partial^2 A(\xi, y)}{\partial\xi^2} \tag{12}$$

with A being the image of a,

$$A(\xi, y) \equiv \int_{\xi}^{\infty} a(\eta, y)\sqrt{\eta-\xi}\,\mathrm{d}\eta = (n_0 - 1)\int_{\xi}^{\infty} \frac{\rho(\eta, y) - \rho_0}{\rho_0}\sqrt{\eta-\xi}\;\mathrm{d}\eta \tag{13}$$

where ρ is the mass density. To apply (10), we identify the Abel image with $F = 2\partial A/\partial\xi$ and corresponding Abel object with $f = 2\partial a/\partial\eta$. Then

$$\frac{\partial a(\eta, y)}{\partial\eta} = \frac{1}{\pi}\frac{\partial^2 G(\eta, y)}{\partial\eta^2} \tag{14}$$

In similar way we invert the second relation in (6) for deflection β in the direction of symmetry axis *y*. In our notation

$$\beta(\xi, y) = \int_{\xi}^{\infty} \frac{\partial a / \partial y}{\sqrt{\eta - \xi}} \, \mathrm{d}\eta = -2 \frac{\partial^2 A(\xi, y)}{\partial \xi \, \partial y} \qquad (15)$$

Comparison with (10) gives the inverse formula

$$\frac{\partial a(\eta, y)}{\partial y} = \frac{2}{\pi} \frac{\partial^2 B(\eta, y)}{\partial \eta^2} \qquad (16)$$

where B represents the Abel image of β. The expressions (6) gives $\partial \alpha / \partial y = \partial \beta / \partial x$ or, equivalently, $\partial g / \partial y = 2 \partial \beta / \partial \xi$. Transition to the Abel images yields $\partial G / \partial y = 2 \partial B / \partial \eta$. The previous formula (16) is then

$$\frac{\partial a(\eta, y)}{\partial y} = \frac{1}{\pi} \frac{\partial^2 G(\eta, y)}{\partial y \, \partial \eta} \qquad (17)$$

System of equations (14,17) together with boundary condition $a \to 0$ for $\eta \to \infty$ has unique solution $a = 1/\pi \, \partial G / \partial \eta$, or, in more detail,

$$\frac{n - n_0}{n} = \frac{1}{\pi} \frac{\partial}{\partial \eta} \int_{\eta}^{\infty} g(\xi, y) \sqrt{\xi - \eta} \, \mathrm{d}\xi \qquad (18)$$

After differentiating and substituting the original quantities $g = \alpha / x$, $\xi = x^2$ and $\eta = r^2$, we obtain the final form of the inversion formula

$$\frac{n(r, y) - n_0}{n_0} = -\frac{1}{\pi} \int_{r}^{\infty} \frac{\alpha(x, y)}{\sqrt{x^2 - r^2}} \mathrm{d}x \qquad (19)$$

The only angular deflections in direction perpendicular to the symmetry axis are needed for the reconstruction of the refractive index.

3.2 Shadow method

Contrary to the schlieren method, the shadow technique is experimentally realized more easily. On the other hand, the quantitative analysis is very questionable, as the changes of illumination depend upon second derivatives and are tenuous. Hence, the mathematical description in terms of Abel transformation is only of theoretical significance.

For cylindrically symmetric distribution of refractive index we get

$$\frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} = p(r, y) \, x^2 + q(r, y) \qquad (20)$$

where

$$p = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial a}{\partial r} \right) = 4 \frac{\partial^2 a}{\partial \eta^2}, \qquad q = \frac{1}{r} \frac{\partial a}{\partial r} + \frac{\partial^2 a}{\partial y^2} = 2 \frac{\partial a}{\partial \eta} + \frac{\partial^2 a}{\partial y^2} \qquad (21)$$

After some routine computations we obtain for the function J from (4)

$$J(\xi, y) = -2\left(4\xi\frac{\partial^3 A}{\partial\xi^3} + 2\frac{\partial^2 A}{\partial\xi^2} + \frac{\partial^3 A}{\partial\xi\,\partial y^2}\right) \tag{22}$$

where A is the image of the relative change of refractive index, a. To reduce the order of derivatives, we introduce the auxiliary function

$$F(\xi, y) = -2\frac{\partial A}{\partial\xi} = \int_{\xi}^{\infty}\frac{a(\eta, y)}{\sqrt{\eta-\xi}}\,\mathrm{d}\eta \tag{23}$$

Then

$$J(\xi, y) = 4\xi\frac{\partial^2 F}{\partial\xi^2} + 2\frac{\partial F}{\partial\xi} + \frac{\partial^2 F}{\partial y^2} \tag{24}$$

Application of the inverse Abel transformation to the formula (23) gives

$$a(\eta, y) = -\frac{1}{\pi}\int_{\eta}^{\infty}\frac{\partial F(\xi, y)/\partial\xi}{\sqrt{\xi-\eta}}\,\mathrm{d}\xi \tag{25}$$

The expression of a is much more complicated than for the schlieren method as it is expressed via the function F, which is connected with the directly measured function J by the partial differential equation (24) of the second order. The theoretical solution of the inversion problem can be schematically expressed as a sequence:

intensity $I \rightarrow J \rightarrow F \rightarrow a \rightarrow$ refractive index n.

4. Numerical modeling

We have developed a set of MATLAB codes, modeling results of schlieren and shadow techniques for arbitrary, axially symmetric distribution of gas density, temperature or pressure. The inverse problem has not yet been solved. The codes are based on the Abel image A of a. Following illustrative examples correspond to the mass density described by

$$\frac{\rho-\rho_0}{\rho_0} = \left[1-\frac{y}{H}\exp\left(1-\frac{y}{H}\right)\right]\,\exp\left[-0.1\left(\frac{r}{R}\right)^2\right] \tag{26}$$

where H and R are the height and radius of the investigated region, respectively. We have chosen these dimensions as $H = R = 20\,\mathrm{cm}$. For atmospheric air at the pressure $p_0 = 1.01325\times10^5\ \mathrm{Pa}$, temperature $T_0 = 293.15\ \mathrm{K}$ and for the light wavelength $\lambda = 632.8\ \mathrm{nm}$ the reference refractive index is $n_0 = 1.0002716$ [6].

Figures 1 and 2 show ray deflections in angular degrees. The angles in our example are very small: The angular deviation α in the direction x perpendicular to the symmetry axis is of order of 0.1°, the deviation β in the direction y is of order of 0.01°. Nevertheless, the experimental arrangement with the diaphragm partly screening the light beam enables to register changes of intensity caused by such small angular deflections.

Figure 3 depicts the relative changes of light intensity, $(I - I_0)/I_0$, obtained by the shadow method. The image is colored as in realistic grey scale looks nearly black. The relative changes are of order of 10^{-4}. These tiny variations together with the dependency of the effect upon second derivatives are the reason while the shadow technique is not suitable for quantitative analysis.

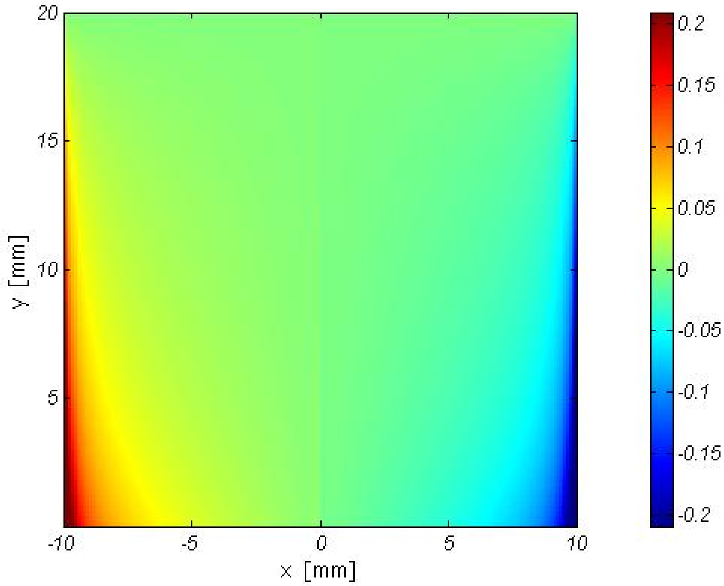

Fig. 1 Schlieren method, $\alpha(x, y)$

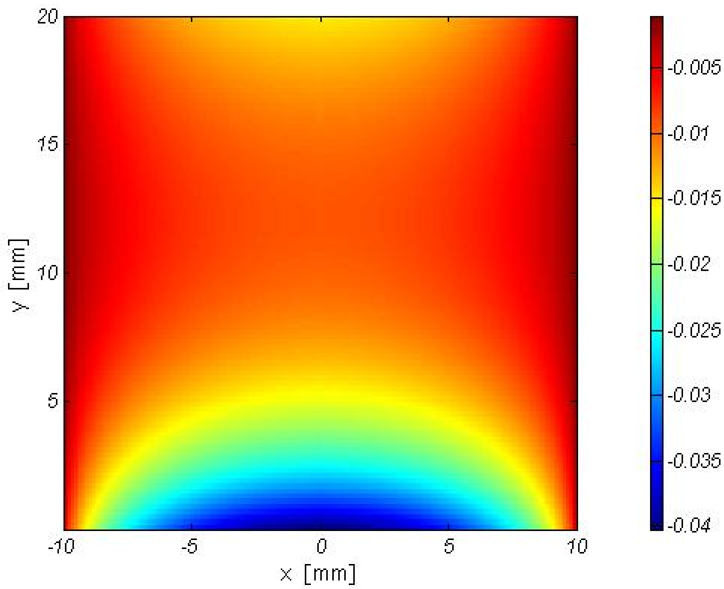

Fig 2 Schlieren method, $\beta(x, y)$

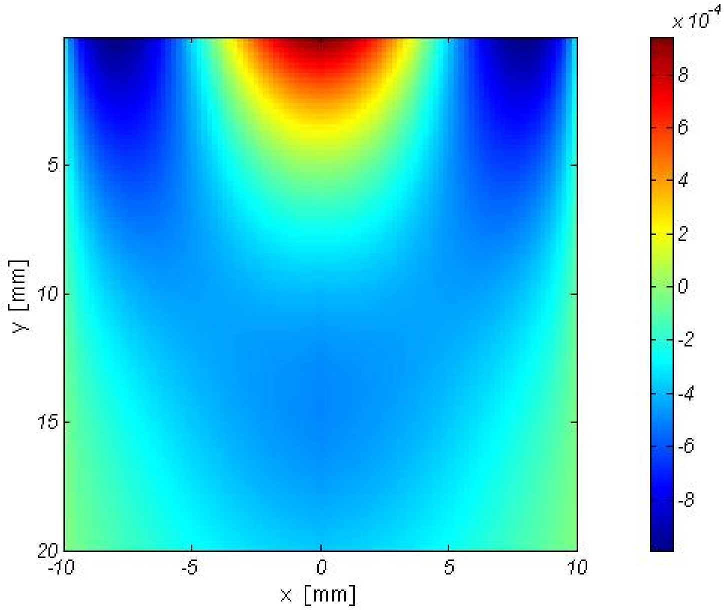

Fig. 3 Shadowgraph, $\Delta I(x,y)/I_0$

5. Conclusion

We have derived formulas concerning images obtained by refractive methods for axially symmetric transparent objects. The formalism of Abel integral transformation can be applied in two ways. First, it enables effective computer modeling of images corresponding to inner conditions given in advance, e.g. as they are theoretically predicted by gas dynamics. The modeled and real images can then be compared and possible optional parameters of the model corrected. Second, developed inversion formula for schlieren method can be directly applied to determine inner distribution of refractive index or mass density.

Together with theoretical solution we have developed several codes written in MATLAB, which for defined distribution of mass density and for given refractive technique or experimental arrangement simulate corresponding measurements. These programs facilitate to interpret various image patterns and estimate the influence of various disturbing effects and random errors.

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