MATHEMATICAL MODELLING OF HUMAN LIMB IN MATLAB

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Abstract

The contribution deals with mathematical modelling of human limb and numerical simulation of mechanical processes taking place during static loadening. It is important to know stress and deformation distributions in lower limb for example for application of the total knee or the total hip replacements. A two dimensional finite element approximation and domain decomposition method are used. System MATLAB, namely toolbox pdetool, is very useful for mesh generating and vizualization.

1 Introduction

In biomechanics of human skeleton there are problems whose investigations lead to solving variational problems. Such problems are described frequently by variational inequalities representing the principle of virtual work in its inequality form. Example of this is bearing of loaded human limb. The hip and the knee joints are the most effortful joints in human body and study of the stress/strain distribution is essential before application of their prosthesis. The success of the artificial replacements depends on many factors. The mechanical factor is an important one. The idea of a prosthesis being a device that transfer the joint loads to the bone allows explaining the mechanical factor in terms of the load transfer mechanism. A complex relation exists between this mechanism, the magnitude direction of the loads, the geometry of the bone-joint prosthesis configuration, the elastic properties of the materials and the physical connections at the material connections. The numerical solution is based on the theory of contact problem in elasticity and the finite element approximation. It is possible to use domain decomposition technique for efficient solution of the problem.

2 The Model

Let the investigated part of the human skeleton occupy a union Ω of "s" bounded domains $\Omega^\iota, \iota = 1, ..., s$ in $\mathbb{R}^2$, denoting separate components of human joints, with Lipschitz boundaries $\partial\Omega^\iota$. Let the boundary $\partial\Omega = \cup_{\iota=1}^{s} \partial\Omega^\iota$ consist of four disjoint parts such that $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c \cup \Gamma_0$. Based on knowledge of the physiological distribution of insertions in bone tissue and skeletal sites through which the loading forces are transmitted due to weight of the human body and due to acting muscle's forces, we eliminate this portion of the skeleton boundary as $\Gamma_\tau = {}^1\Gamma_\tau \cup {}^2\Gamma_\tau$. By Γ_u we denote the part of the skeletal boundary, where we simulate its fixation or the surgical osteotomy technique, respectively. The common contact boundary between both joint components Ω^k and Ω^l before deformation we denote by $\Gamma_c^{kl} = \partial\Omega^k \cap \partial\Omega^l$, $k, l = 1, ..., s$, $k \neq l$, and by $\Gamma_c = \cup_{k,l} \Gamma_c^{kl}$ the whole contact boundary. Moreover, the boundary Γ_0 simulates the symmetry of the pelvis. Let body forces $\mathbf{F}$, surface tractions $\mathbf{P}$ and slip limits g^{kl} be given.

We have the following problem ($\mathcal{P}$): find the displacements $\mathbf{u}^\iota$ in all Ω^ι such that

$$\frac{\partial}{\partial x_j} \tau_{ij}(\mathbf{u}^\iota) + F_i^\iota = 0 \quad \text{in } \Omega^\iota,\ 1 \leq \iota \leq s,\ i = 1, 2, \tag{1}$$

where the stress tensor τ_{ij} is defined by

$$\tau_{ij}(\mathbf{u}^\iota) = c_{ijkl}^\iota e_{kl}(\mathbf{u}^\iota) \quad \text{in } \Omega^\iota,\ 1 \leq \iota \leq s,\ i = 1, 2, \tag{2}$$

with boundary conditions

$$\tau_{ij}(\mathbf{u},T)n_j = P_i \quad \text{on } {}^1\Gamma_\tau,\ i=1,2, \tag{3}$$

$$\tau_{ij}(\mathbf{u},T)n_j = 0 \quad \text{on } {}^2\Gamma_\tau,\ i=1,2, \tag{4}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on } \Gamma_u, \tag{5}$$

$$u_n^k - u_n^l \le 0,\ \tau_n^k \le 0,\ (u_n^k - u_n^l)\tau_n^k = 0 \quad \text{on } \cup_{k,l}\Gamma_c^{kl},\ 1 \le k,l \le s, \tag{6}$$

$$\left.\begin{array}{l} |\ \tau_t^{kl}\ | \le g^{kl} \quad \text{on } \cup_{k,l}\Gamma_c^{kl},\ 1 \le k,l \le s, \\ |\ \tau_t^{kl}\ | < g^{kl} \implies \mathbf{u}_t^k - \mathbf{u}_t^l = 0, \\ |\ \tau_t^{kl}\ | = g^{kl} \implies \text{there exists } \vartheta \ge 0 \text{ such that } \mathbf{u}_t^k - \mathbf{u}_t^l = -\vartheta\tau_t^{kl}, \end{array}\right\} \tag{7}$$

$$u_n = 0,\ \tau_t = 0 \quad \text{on } \Gamma_0. \tag{8}$$

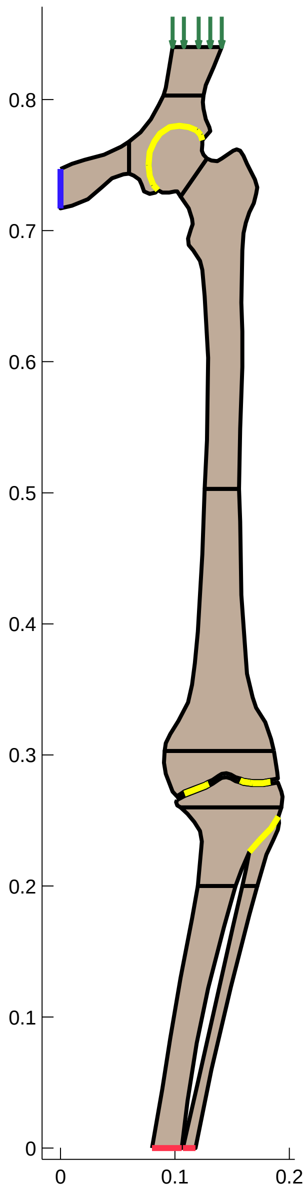


Figure 1: The model of human limb

Here $e_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the small strain tensor, normal and tangential components of displacement ($\mathbf{u} = (u_i), i = 1,2$) and stress ($\tau = (\tau_i)$) vectors $u_n^k = u_i^k n_i^k$, $u_n^l = u_i^l n_i^l = -u_i^k n_i^k$ (no sum over k or l), $\mathbf{u}_t^k = (u_{ti}^k)$, $u_{ti}^k = u_i^k - u_n^k n_i^k$, $\mathbf{u}_t^l = (u_{ti}^l)$, $u_{ti}^l = u_i^l - u_n^l n_i^l$, $i = 1,2$, $\tau_n^k = \tau_{ij}^k n_i^k n_j^k$, $\tau_t^k = (\tau_{ti}^k)$, $\tau_{ti}^k = \tau_{ij}^k n_j^k - \tau_n^k n_i^k$, $\tau_n^l = \tau_{ij}^l n_i^l n_j^l$, $\tau_t^l = (\tau_{ti}^l)$, $\tau_{ti}^l = \tau_{ij}^l n_j^l - \tau_n^l n_i^l$, $\tau_t^{kl} \equiv \tau_t^k$. Assume that c_{ijkl}^ι are positive definite symmetric matrices such that $0 < c_0^\iota \le c_{ijkl}^\iota \xi_{ij}\xi_{kl} \mid \xi \mid^{-2} \le c_1^\iota < +\infty$ for a.a. $\mathbf{x} \in \Omega^\iota, \xi \in \mathbb{R}^4, \xi_{ij} = \xi_{ij}$, where c_0^ι, c_1^ι are constants independent of $\mathbf{x} \in \Omega^\iota$.

Let us introduce $W = \sqcap_{\iota=1}^s [H^1(\Omega^\iota)]^2$, $\|\mathbf{v}\|_W = (\sum_{\iota\le s}\sum_{i\le 2}\|v_i^\iota\|_{1,\Omega^\iota}^2)^{\frac{1}{2}}$ and the sets of virtual and admissible displacements $V_0 = \{\mathbf{v} \in W \mid \mathbf{v} = 0 \text{ on } \Gamma_u \text{ and } v_n = 0 \text{ on } \Gamma_0\}$, $V = \mathbf{u}_0 + V_0$, $K = \{\mathbf{v} \in V \mid v_n^k - v_n^l \le 0 \text{ on } \cup_{k,l}\Gamma_c^{kl}\}$. Assume that $u_{0n}^k - u_{0n}^l = 0$ on $\cup_{k,l}\Gamma_c^{kl}$. Let $c_{ijkl}^\iota \in L^\infty(\Omega^\iota)$, $F_i^\iota \in L^2(\Omega^\iota)$, $P_i \in L^2({}^1\Gamma_\tau)$, $\mathbf{u}_0^\iota \in [H^1(\Omega^\iota)]^2$.

Multiplying equation (1) by a test function, integrating per partes over the domain Ω^ι and using the boundary conditions and assuming that $\mathbf{u}_0$ satisfies conditions $u_{0n}^k - u_{0n}^l = 0$ on $\cup_{k,l}\Gamma_c^{kl}$, we obtain the following variational problem:

Definition

We said that the function $\mathbf{u}$ *is a weak solution of the problem* ($\mathcal{P}$), *if* $\mathbf{u} - \mathbf{u}_0 \in K$ *and*

$$a(\mathbf{u},\mathbf{v}-\mathbf{u}) + j_g(\mathbf{v}) - j_g(\mathbf{u}) \ge L(\mathbf{v}-\mathbf{u}) \qquad \forall \mathbf{v} \in K, \tag{9}$$

where

$$\left.\begin{array}{l} a(\mathbf{u},\mathbf{v}) = \sum_{\iota=1}^s \int_{\Omega^\iota} c_{ijkl}^\iota e_{ij}(\mathbf{u}^\iota)e_{kl}(\mathbf{v}^\iota)\,d\mathbf{x}, \\ j_g(\mathbf{v}) = \sum_{k,l}\int_{\Gamma^{kl}} g^{kl} \mid \mathbf{v}_t^k - \mathbf{v}_t^l \mid ds, \\ L(\mathbf{v}) = \sum_{\iota=1}^s \int_{\Omega^\iota} F_i^\iota v_i^\iota d\mathbf{x} - \sum_{\iota\le s}\int_{\Gamma_\tau^\iota} P_i^\iota v_i^\iota\,ds. \end{array}\right\} \tag{10}$$

The model of human limb is presented in Fig. 1.

3 Numerical results

The model of the human limb was derived from the X-ray image. In the model the material parameters are as follows: Young's modulus for bone $E = 1,71 \times 10^{10}$ [Pa] and for cartilage $E = 0,492 \times 10^{9}$ [Pa]. Poisson's ratio for bone $\nu = 0,25$ and for cartilage $\nu = 0,1$.

On the part of the boundary of the investigated skeleton denoted by red color (see Fig. 1) the "stick" condition (5) is prescribed. Conditions (6) and (7) holds for the contact boundaries which are denoted by yellow color. Condition (8) simulates the symmetry of the pelvis and we assume that it's prescribed on blue part of boundary. The pelvis is loaded on the upper part of boundary (used green color) by a loading 1×10^{6} [Pa]. The loadings evoked by muscular forces were neglected.

The numerical solution is based on the finite element method and domain decomposition technique. Using of the finite element approximation is discussed in [3]. For mesh generating we used toolbox `pdetool` - the partial differential equation toolbox. For efficient solution of the problem we used domain decomposition algorithm which is described in [1] and [2].

Whole area is splitted into 16 subdomains. Discretization statistics are characterized by 1436 nodes and 2137 elements, 272 interface elements between subdomains of domain decomposition.

For vizualization of obtained solutions we exploited again toolbox `pdetool`, mainly function `pdeplot`.

In Fig. 2 the deformation of the limb after loading is presented. Details of the deformations are shown on Fig. 3(a) for the hip joint and on Fig. 3(b) for the knee joint.

On Figs 4(a) and 4(b) the horizontal and the vertical components of the displacement are presented.

In Figs 5(a) and 5(b) the details of the principal stresses for the hip joint and for the knee joint are presented.

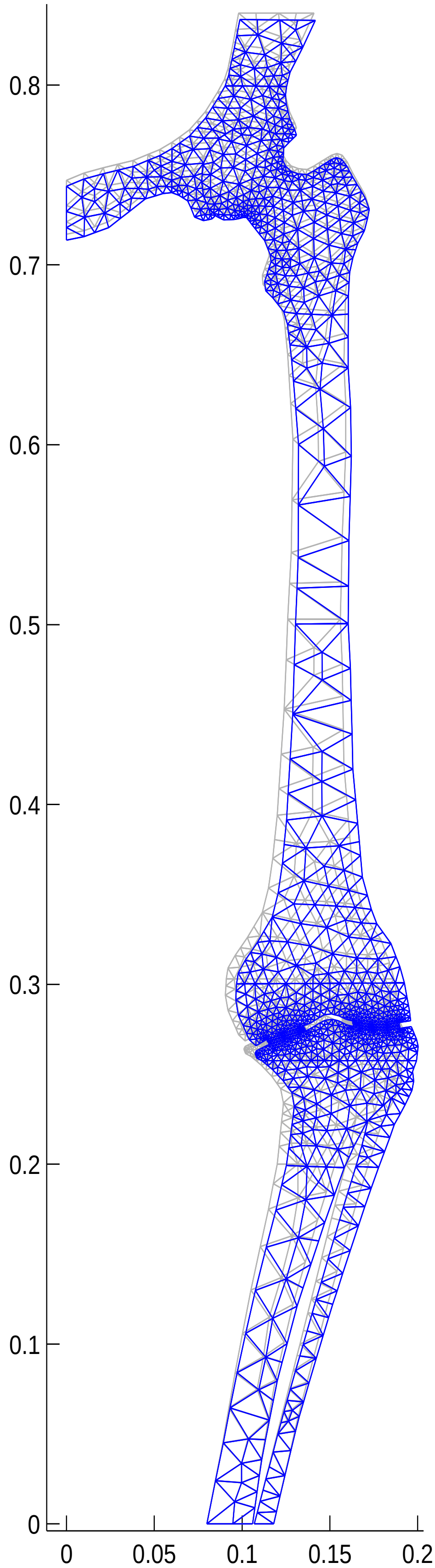
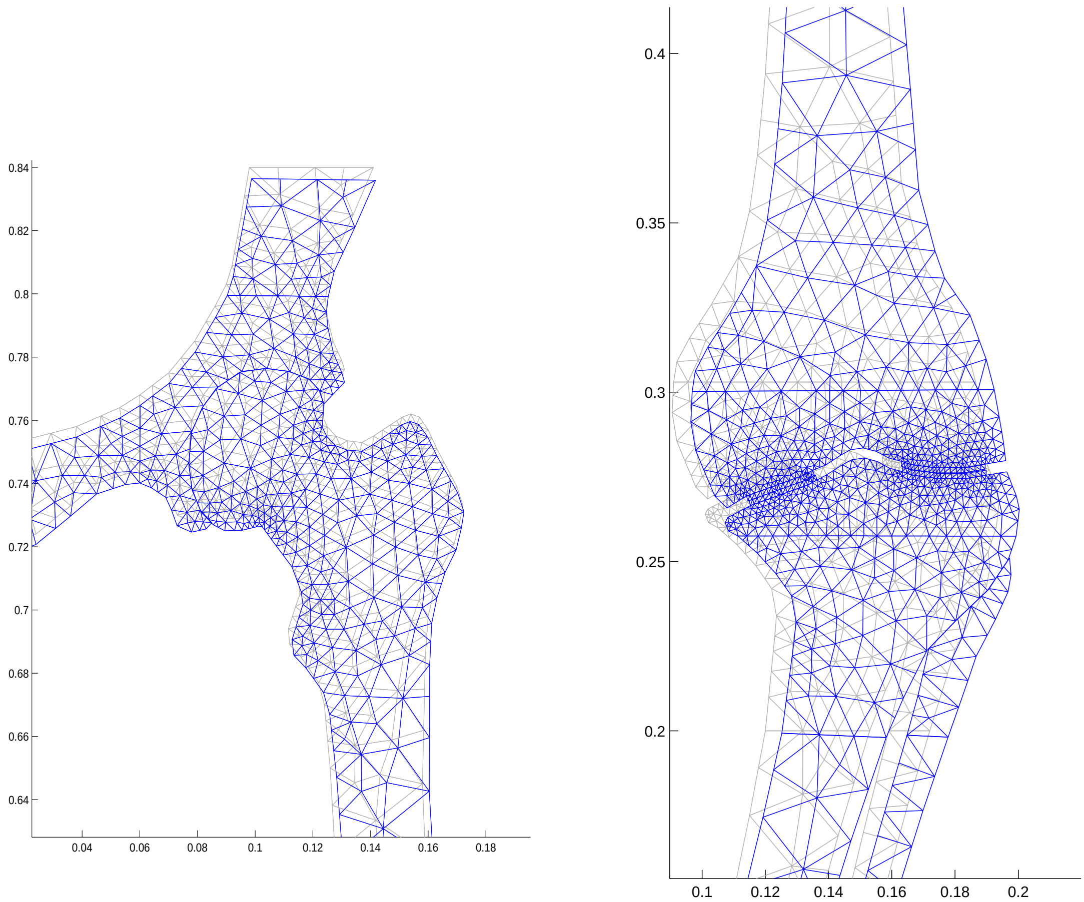


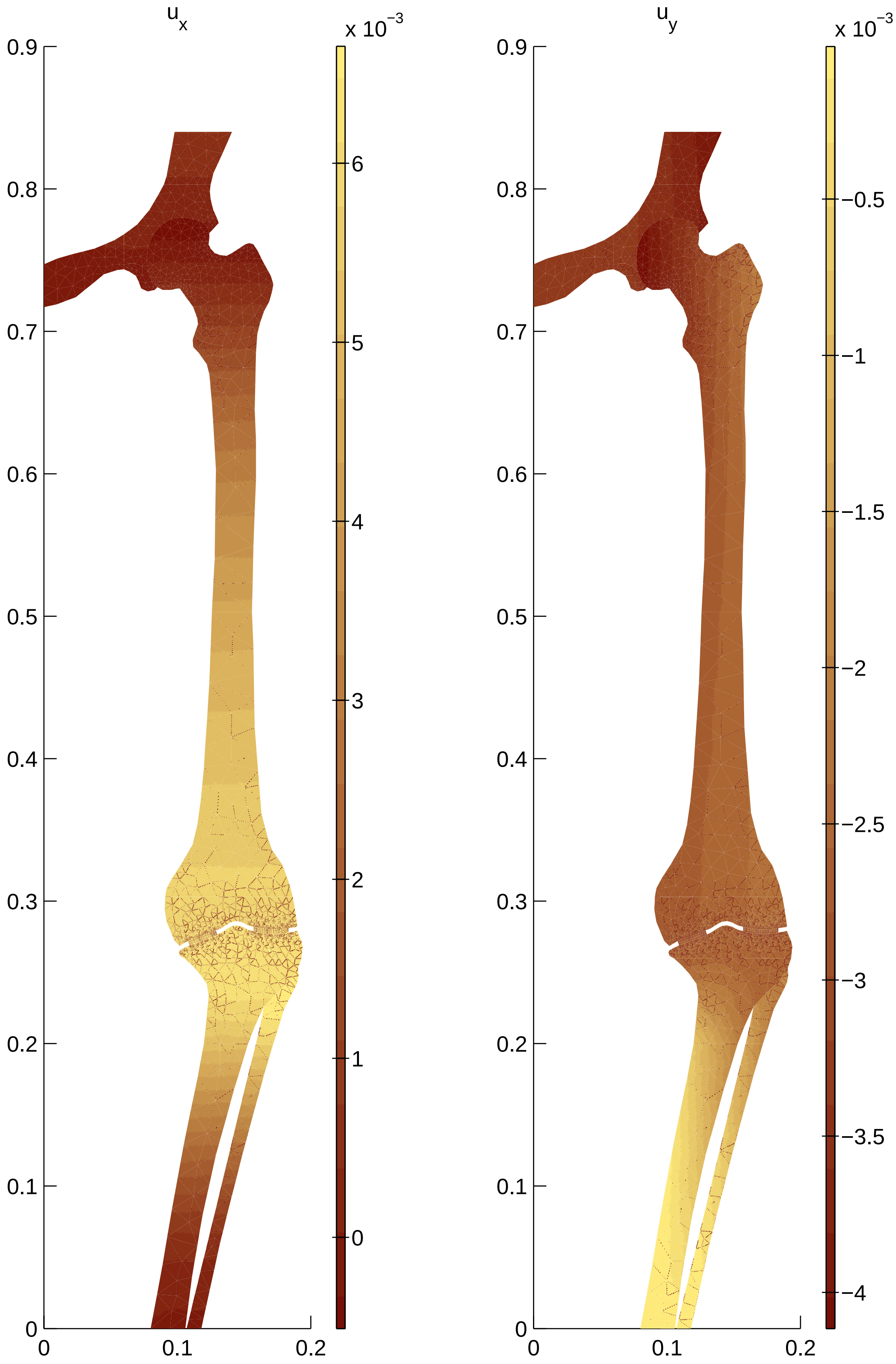
Figure 2: Deformations after static loadening



(a) The hip joint

(b) The knee joint

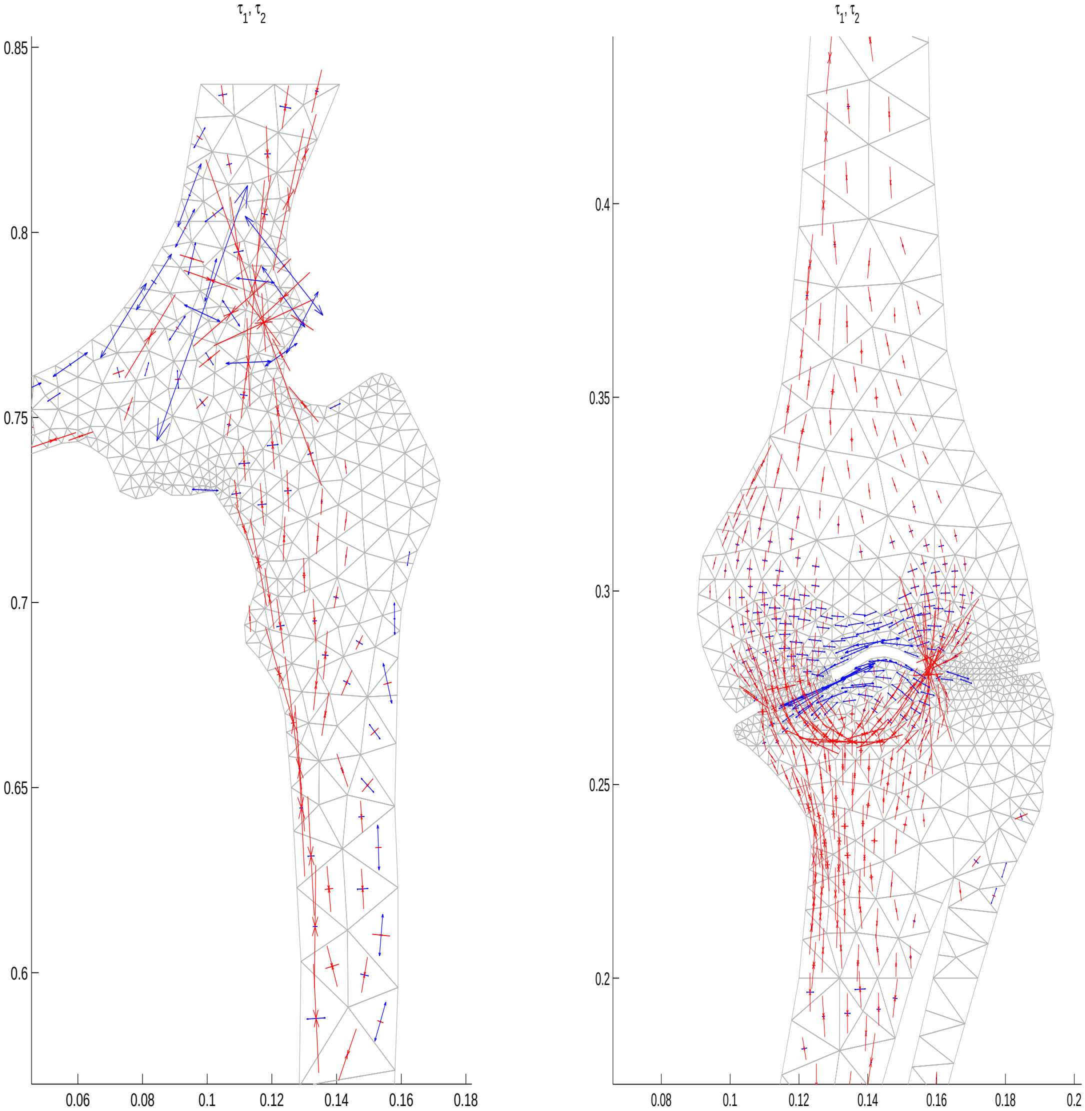
Figure 3: Detail of deformations



(a) Horizontal component

(b) Vertical component

Figure 4: The displacement



(a) The hip joint

(b) The knee joint

Figure 5: Detail of the principal stresses

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