AML PRECONDITIONING OF COARSE PROBLEMS OF THE BDDC METHOD

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Abstract

We study preconditioning of a coarse problem within a special domain decomposition algorithm for solving partial differential equations. We consider the method of balanced domain decomposition by constraints where coarse base functions are defined by some nodal values on interfaces of subdomains and fulfil a minimal energy condition. An algebraic multilevel preconditioning strategy is used for the coarse problem. Numerical estimates of the main characteristic of the quality of preconditioning are presented.

1 A method of BDDC

A method of balanced domain decomposition by constraints (BDDC) is an iterative algorithm for numerical solution of elliptic partial differential equations discretized by finite element (FE) method which exploits a nonoverlaping partition of a domain [3]. In each iteration, the problem is restricted on every particular subdomain and solved, and a certain coarse grid solution is found on the whole domain. Base functions for the coarse problem are defined by nodal values (degrees of freedom, DOFs) in some points on interfaces of subdomains and have minimal energy on each subdomain and null normal derivatives on all interfaces, see two samples on Figure 1. Then in general, the coarse functions are discontinuous on interfaces of subdomains up to the nodes where coarse DOFs are defined . The coarse problem itself can be large and often is a bottleneck of BDDC computation and thus an appropriate preconditioning is desired [2].

2 AML preconditioning

In our contribution we present a new strategy of preconditioning of the coarse problem of BDDC. This is based on an algebraic multilevel (AML) preconditioning technique [1]. In spite of classical application of AML preconditioning directly to finite element bases, we utilize a hierarchical splitting of the coarse space of BDDC. A quality of AML preconditioning can be measured by

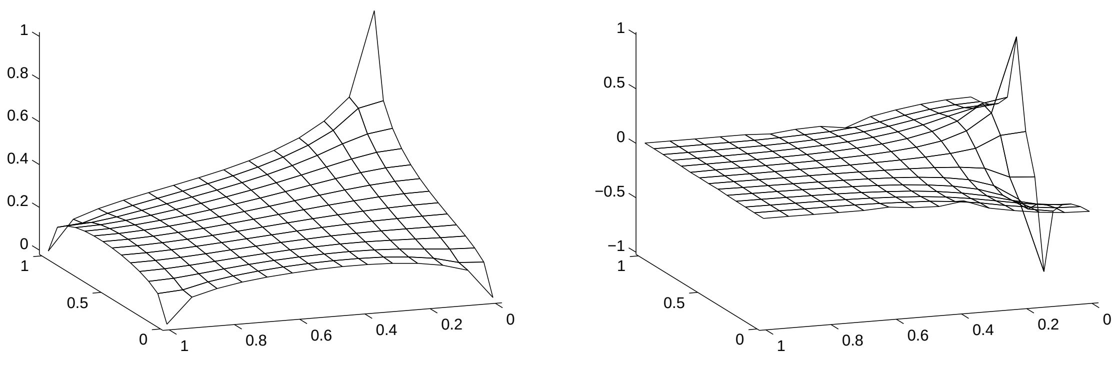

Figure 1: Two samples of coarse base functions restricted on a single subdomain defined by different sets of coarse DOFs: four corner nodal values (left) and eight boundary nodal values (right).

a constant γ in the strengthened Cauchy-Buniakowski-Schwarz (CBS) inequality [1]. After block diagonal preconditioning with two blocks corresponding to stiffness matrices of two hierarchical base systems, a resulting condition number is

$$\kappa = \frac{1+\gamma}{1-\gamma}.$$

As a result, our wish is to find a hierarchical splitting with a sufficiently low CBS constant.

3 Numerical estimates of CBS constants for hierarchical coarse spaces

We provide numerical estimates of the CBS constants for several two- and three-dimensional problems. An equation of diffusion and a linear elasticity equation are considered. For discretisation, bilinear or trilinear FEs are used with a rectangular or prismatic support, respectively. The subdomains are of a rectangular or prismatic shape as well. Coarse base functions are defined by corner nodal values on subdomains. In every problem a hierarchical splitting of the coarse base system is constructed with coefficients which are equal to that for hierarchical transformation of FEs [4]. In each test we specify only a bilinear form $a(\cdot,\cdot)$ of the weak formulation of the problem since neither boundary conditions nor a right hand side of the equation influence estimates of the CBS constant γ.

The estimate of γ can be calculated from exploiting the properties of the coarse function on a single subdomain only. A mesh of a reference subdomain may influence the value of γ. Then in each graph, a number of elements in a reference subdomain is indicated. Our main interest is to examine a behavior of γ when varying the coefficients of a bilinear form $a(\cdot,\cdot)$. All computations are performed in Matlab.

Diffusion equation (D2). Bilinear form $a(\cdot,\cdot)$ has the form

$$a(u,v) = \int_\Omega (\nabla u)^T C \nabla v \,\mathrm{d}x,$$

where

$$C = \begin{pmatrix} 1 & c \\ c & d \end{pmatrix}.$$

The values of c and d are assumed to be constant on each subdomain.

Let us compare the estimates of γ^2 for the case $c = 0$ and $d \in (0,1\rangle$ in Figure 2 on the left for meshes of a reference subdomain 1×1, 5×5 and 50×50, respectively. On the right, values of γ^2 are presented for $d = 1$ and $c \in (-1, 0\rangle$.

Diffusion equation (D3). Bilinear form $a(\cdot,\cdot)$ is

$$a(u,v) = \int_\Omega (\nabla u)^T C \nabla v \,\mathrm{d}x,$$

where

$$C = \begin{pmatrix} 1 & c_{12} & c_{13} \\ c_{12} & d_2 & c_{23} \\ c_{13} & c_{23} & d_3 \end{pmatrix},$$

such that d_i and c_{ij} are constant on subdomains and matrix C is positive definite.

The estimates of γ^2 for the case $d_2 = 1$, $c_{ij} = 0$ and $d_3 \in (0,1\rangle$ are shown in the left part of Figure 3 for meshes of a reference prismatic subdomain $1\times 1\times 1$, $3\times 3\times 3$ and $10\times 10\times 10$, respectively. In the right hand side of Figure 3, values of γ^2 for the case $d_i = 1$ and $c_{12} = c_{13} = c_{23} \in (-0.5, 0\rangle$ are presented.

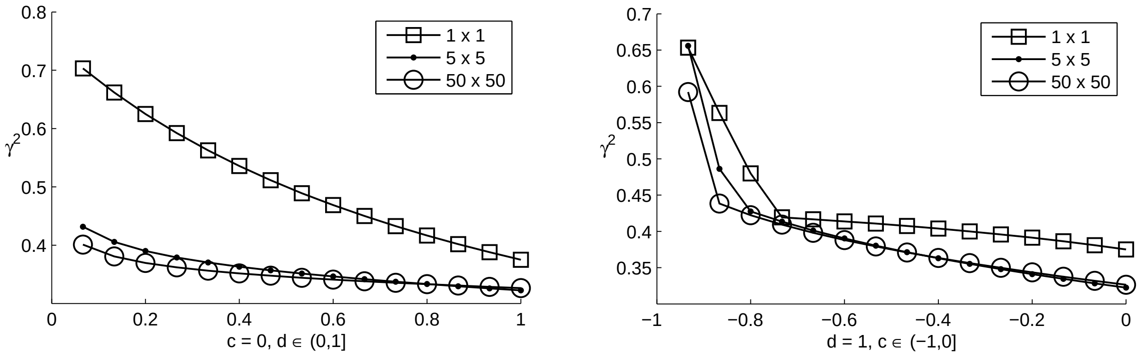

Figure 2: Numerical estimates of γ^2 for hierarchical splitting of coarse problems of BDDC for equation D2. Constant off-diagonal coefficient $c = 0$ and varying $d \in (0, 1\rangle$ (left), and the case $d = 1$ and $c \in (-1, 0\rangle$ (right). Each set of values is calculated for three different meshes of a reference subdomain.

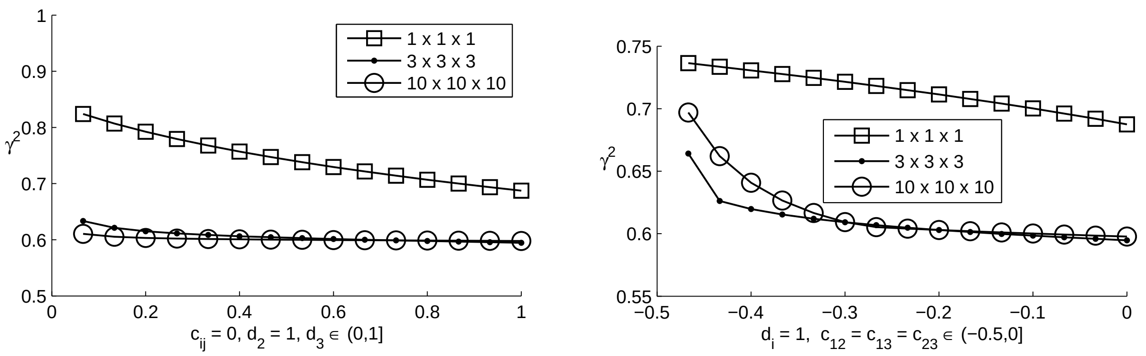

Figure 3: Numerical estimates of γ^2 for hierarchical splitting of coarse problems of BDDC for equation D3. Diagonal coefficient $d_2 = 1$ and off-diagonal coefficients $c_{ij} = 0$ and varying $d_3 \in (0, 1\rangle$ (left), and the case $d_2 = d_3 = 1$ and $c_{12} = c_{13} = c_{23} \in (-0.5, 0\rangle$ (right) for three different meshes of a reference subdomain.

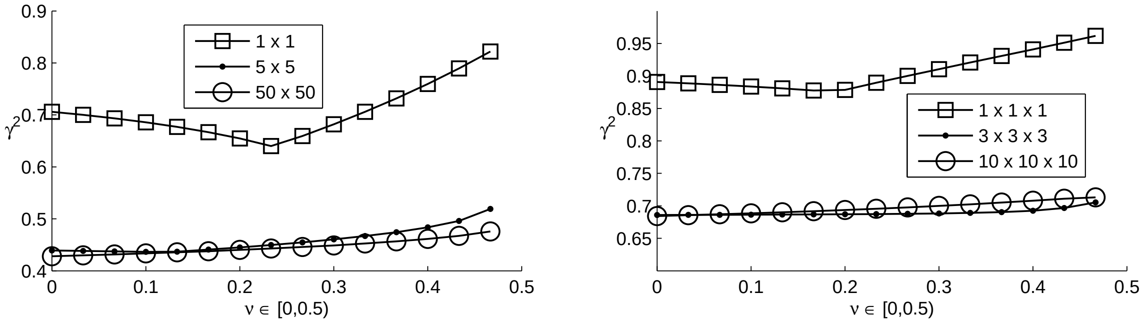

Figure 4: Numerical estimates of γ^2 for hierarchical splitting of coarse problems of BDDC for equations E2 (left) and E3 (right). Varying Poisson ratio $\nu \in \langle 0, 0.5)$ and three different meshes of reference subdomains.

Elasticity equations (E2) and (E3). Bilinear form $a(\cdot,\cdot)$ is

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega 2\mu(\nabla^{(s)}\mathbf{u})^T\nabla^{(s)}\mathbf{v} + \lambda\,\mathrm{div}\,\mathbf{u}\cdot\mathrm{div}\,\mathbf{v}\,\mathrm{d}x,$$

where $\nabla^{(s)}\mathbf{u} = \varepsilon(\mathbf{u})$ and

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The constant μ is the share modulus of a material and λ is the Lamé constant. Substituting $\mu = E/2/(1+\nu)$ and $\lambda = E\nu/(1+\nu)/(1-2\nu)$ we can use the Poisson ratio ν and the modulus of elasticity E. This problem is considered as a two- or three-dimensional problem, respectively.

The stimates of γ^2 for elasticity equations and for $\nu \in \langle 0, 0.5)$ can be found in Figure 4. A two-dimensional case for partition of a rectangular subdomain 1×1, 5×5 and 50×50, respectively, is shown on the left. A three-dimensional case for meshes $1\times 1\times 1$, $3\times 3\times 3$ and $10\times 10\times 10$, respectively, on a reference prismatic subdomain can be found in the right part of Figure 4.

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