THE MEAN-VARIANCE MODEL

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Abstract

The mean-variance model has been a simple but quite intuitive framework. The framework helps to build efficient portfolios of investment assets. Within the framework 3-stock portfolios were constructed to test the mean-variance strategy. The selected equity portfolios have shown the mean-variance superiority over the plain benchmark index.

1 Introduction

Portfolio selection problems [1] were formulated for the first time by Markowitz [3]. They consist of allocating capital over a number of available assets in order to maximize the return on the investment while minimizing the risk using mathematical techniques. In the proposed models, the return is measured by the expected value of the random portfolio return, while the risk is quantified by the variance of the portfolio (a Mean-Variance Model).

2 Background

Consider n risky assets [1] that can be chosen by an investor in the financial market. Let $\mathbf{r} = (r_1, \ldots, r_n)^T \in \Re^n$ denote the uncertain returns of the n risky assets from the current time $t = 0$ to a fixed future time $t = T$. Let $\mathbf{x} = (x_1, \ldots, x_n)^T \in \Re^n$ denote the percentage of the available funds to be allocated in each of the n risky assets. A *portfolio allocation model* aims at finding the optimal (best) portfolio $\mathbf{x}$ to be constructed at $t = 0$, in order to maximize the portfolio's future return $\mathbf{r}^T\mathbf{x}$ from $t = 0$ to $t = T$.

We consider in a one-period portfolio selection problem [1, 3]. Let the random vector $\mathbf{r} = (r_1, \ldots, r_n) \in \Re^n$ denote random returns of the n risky assets, and $\mathbf{x} = (x_1, \ldots, x_n)^T \in X = \{\mathbf{x} \in \Re^n : \mathbf{1}_n{}^T\mathbf{x} = 1\}$ denote the proportion of the portfolio to be invested in the n risky assets, where T means transposition and $\mathbf{1}_n$ denotes a vector of all ones. Suppose that $\mathbf{r}$ has a probability distribution $p(\mathbf{r})$ with mean vector μ and covariance matrix Σ. Then the target of the investor is to choose an optimal portfolio $\mathbf{x}$ that lies on the mean-risk efficient frontier. In the *Markowitz model* [3], the mean of a portfolio is defined as the expected value of the portfolio return, $\mu^T\mathbf{x}$, and the risk is defined as the variance of the portfolio return, $\mathbf{x}^T\Sigma\mathbf{x}$.

Mathematically, minimizing the variance subject to target and budget constraints leads to a formulation like:

$$\min_{\mathbf{x}} \left\{ \mathbf{x}^T\Sigma\mathbf{x} : \mu^T\mathbf{x} \geq \mu_0, \mathbf{1}_n{}^T\mathbf{x} = 1 \right\}, \tag{1}$$

where μ_0 is the minimum expected return. There are two implicit assumptions in this formulation: i) the first two moments of portfolio return exist and ii) the initial wealth is normalized to be 1 without loss of generality.

3 Empirical Verification

The section gives an overview of data, methodology and software involved in the experiments as a possible verification of the theoretical models.

3.1 Data

The primary stock selection was narrowed to the U.S. traded stocks (Dow 30, S&P 500) and ETFs due to long time series and data availability. Experimental data were retrieved from financial sources. The data related to stocks and ETFs were from the financial portal website [`http://finance.yahoo.com`]. The statistics such as variances, means and covariances of stocks, ETFs and indices were calculated from the historical time series (1st Jan 2010 - 22nd Oct 2013). We assumed that these statistics were close approximations of current values of the selected assets.

3.2 Methodology

The tested portfolios were the equity based portfolios. We concentrated on simple three-stock portfolios. The benchmark as such the U. S. large cap index (S&P 500) was selected for tracking purposes.

The tested portfolios were of three types: (i) long-only equity, (ii) long-short equity and (iii) long-short equity with leverage (due to the Kelly criterion [5]). The long equity side was constructed from subsets of carefully selected equities or equity ETFs. The short side was achieved via short-selling stocks or short-selling equity market indexes.

There have been many derivatives available for hedging purposes such as options, warrants, futures and leveraged ETF (Exchange Traded Funds). We assumed that we made do without them during experiments, but they may have improved the risk-adjusted profile if needed.

The tested portfolios were leveraged from zero up to 100% (200% total exposure, but still under-betting the Kelly criterion [4]).

3.3 Experiments

We experimented with three unnamed stocks (ETFs) (S_1, S_2, S_3). In Table 1, there are expected returns (μ) and standard deviations (σ) for the stocks and the S&P 500 index.

	μ	σ
S_1	0.1000	0.0707
S_2	0.1500	0.2000
S_3	0.1200	0.1517
$sp500$	0.1152	0.0604

Table 1: (μ, σ) pairs. There are expected returns (μ) and standard deviations (σ) for the stocks and the S&P 500 index.

In the matrices ($Corr$, Cov), there were expected correlations and covariances between three selected stocks.

$$\mathbf{Corr} = \begin{pmatrix} 1.0000 & -0.7071 & 0.3730 \\ -0.7071 & 1.0000 & -0.0659 \\ 0.3730 & -0.0659 & 1.0000 \end{pmatrix}$$

$$\mathbf{Cov} = \begin{pmatrix} 0.005 & -0.010 & 0.004 \\ -0.010 & 0.040 & -0.002 \\ 0.004 & -0.002 & 0.023 \end{pmatrix}$$

3.4 Results

Here, we describe our results on the Mean-Variance Model. The results are presented in Figures 1, 2, 3 and 4.

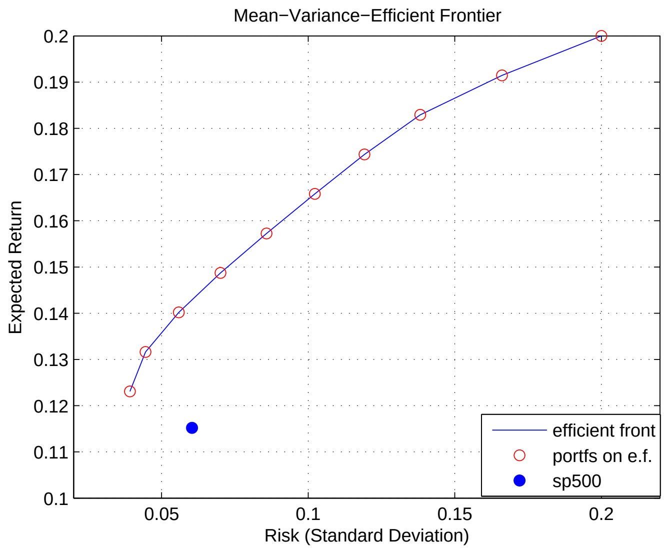

Figure 1: Mean-Variance-Efficient Frontier. The efficient frontier and 10 risky portfolios on the efficient frontier. Portfolios weights are in Table 2.

S_1	S_2	S_3
0.7692	0.2308	0.0000
0.6667	0.2991	0.0342
0.5443	0.3478	0.1079
0.4220	0.3964	0.1816
0.2997	0.4450	0.2553
0.1774	0.4936	0.3290
0.0550	0.5422	0.4027
0	0.6581	0.3419
0	0.8291	0.1709
0	1.0000	0

Table 2: Portfolios weights and risk. These are calculated portfolio weights related to Figure 1. No short-selling was allowed. The lowest risk portfolio splits weights between S_1 (0.7692) and S_2 (0.2308). The highest risk portfolio concentrates in one stock S_2 (1.0000).

The presented research provides a simple procedure how to construct, optimize and balance the equity based portfolios. The calculated statistics and Figures were accomplished in Matlab R2011a (7.12.0) software. The current setup provides far more functionality than was published. E.g. splitting stocks into industry groups, restricting on local and global regions, other limiting and restrictive constrains might be declared on the model.

4 Discussion

We experimented with the simple optimization portfolio strategy (μ,σ). The results were presented in Figures and appropriately commented. The achieved results reflected the reality and

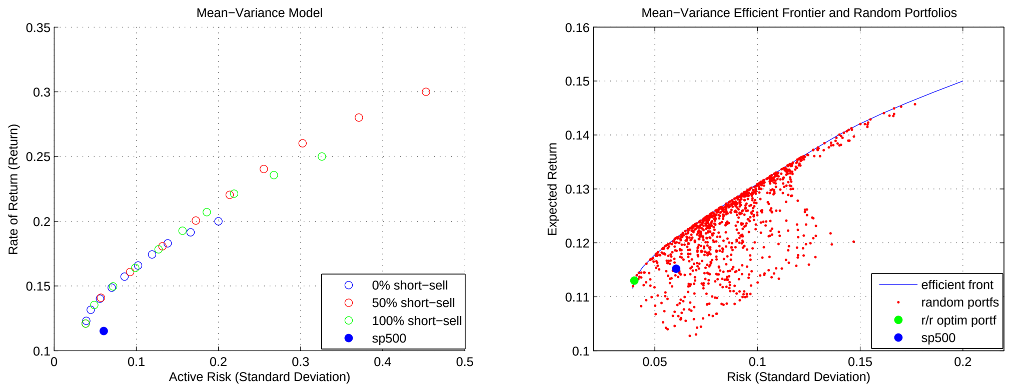

Figure 2: Mean-Variance-Efficient Frontier. Three types of portfolios with different levels of short-selling (0%, 50% and 100%, left). Mean-Variance-Efficient Frontier and 1000 randomly generated portfolios (right).

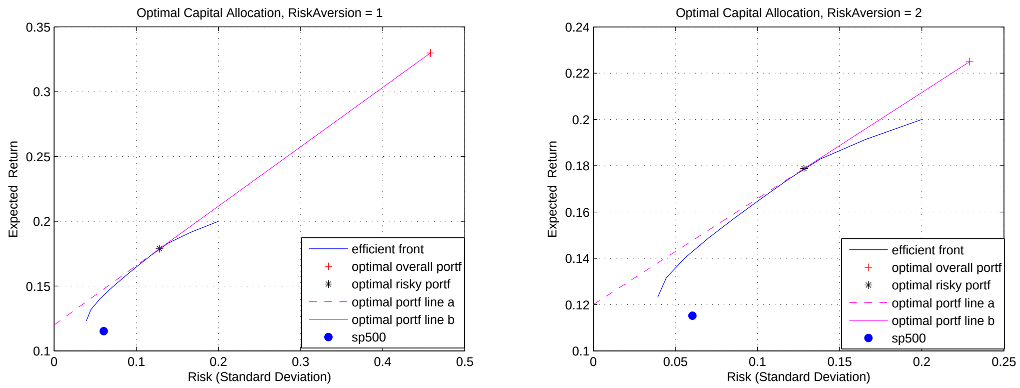

Figure 3: Optimal Capital Allocation. The differences of risk aversion (RA) in the presented setups such as RA = 1 (left) and RA = 2 (right).

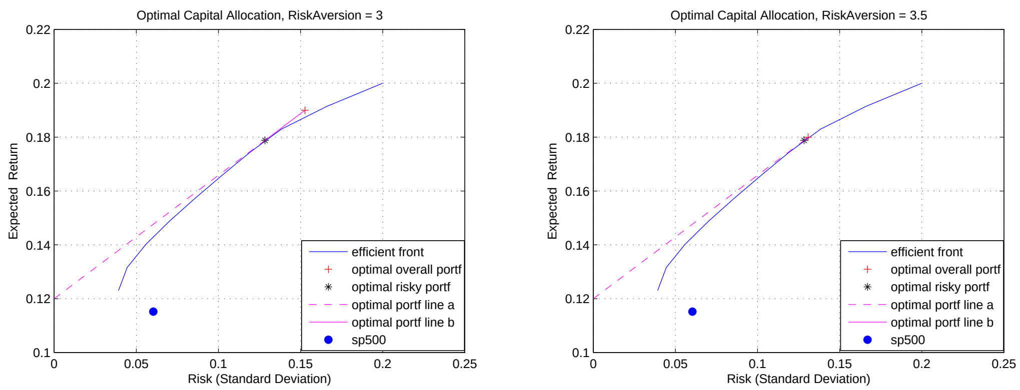

Figure 4: Optimal Capital Allocation. The differences of risk aversion (RA) in the presented setups such as RA = 3 (left) and RA = 3.5 (right).

sanity-testing. The experiments showed the relation between risk and return. As mentioned, there were some hidden flaws in the model (such as descriptive statistics of assets fluctuate) but nevertheless this model worked consistently.

There were no discussion about the relevance to capital markets, allocation strategies, nor real-life portfolio construction. Some parts of applied statistics [2] and econometrics [6] might have needed more attention. The presented work was about understanding the theory, applied statistics and simulation software. The experiments with extended stress-testing, re-allocations and optimizations were not included.

5 Conclusion

The experiments were related to (μ, σ) equity based portfolios. These are portfolios that provide limited downside risks, low and short draw-downs and substantial growth prospects. The portfolios are constructed from two asset classes (equity, cash). One can conclude that:

- The mean-variance framework was introduced. The mean-variance model provides a simple but quite intuitive framework for the portfolio construction.
- The short-selling technique extends expected returns as well as risks of the portfolios.
- The mean-variance strategy was tested with 3-stock portfolios. The stock portfolios shown the mean-variance superiority over the benchmark index.

The (μ,σ) strategy was tested against the capital market data from the period 2010-2013. The tests showed the diversification and optimization effects related to individual securities against the equity index. The robust strategies and portfolios deliver high return-risk ratios. This means, in our case, out-performance of the S&P 500 index with low volatility and robustness to market shocks and other external macro events.

References

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